Lecture: P1_Wk1_L5

Inter-Molecular Forces: Keesom Force

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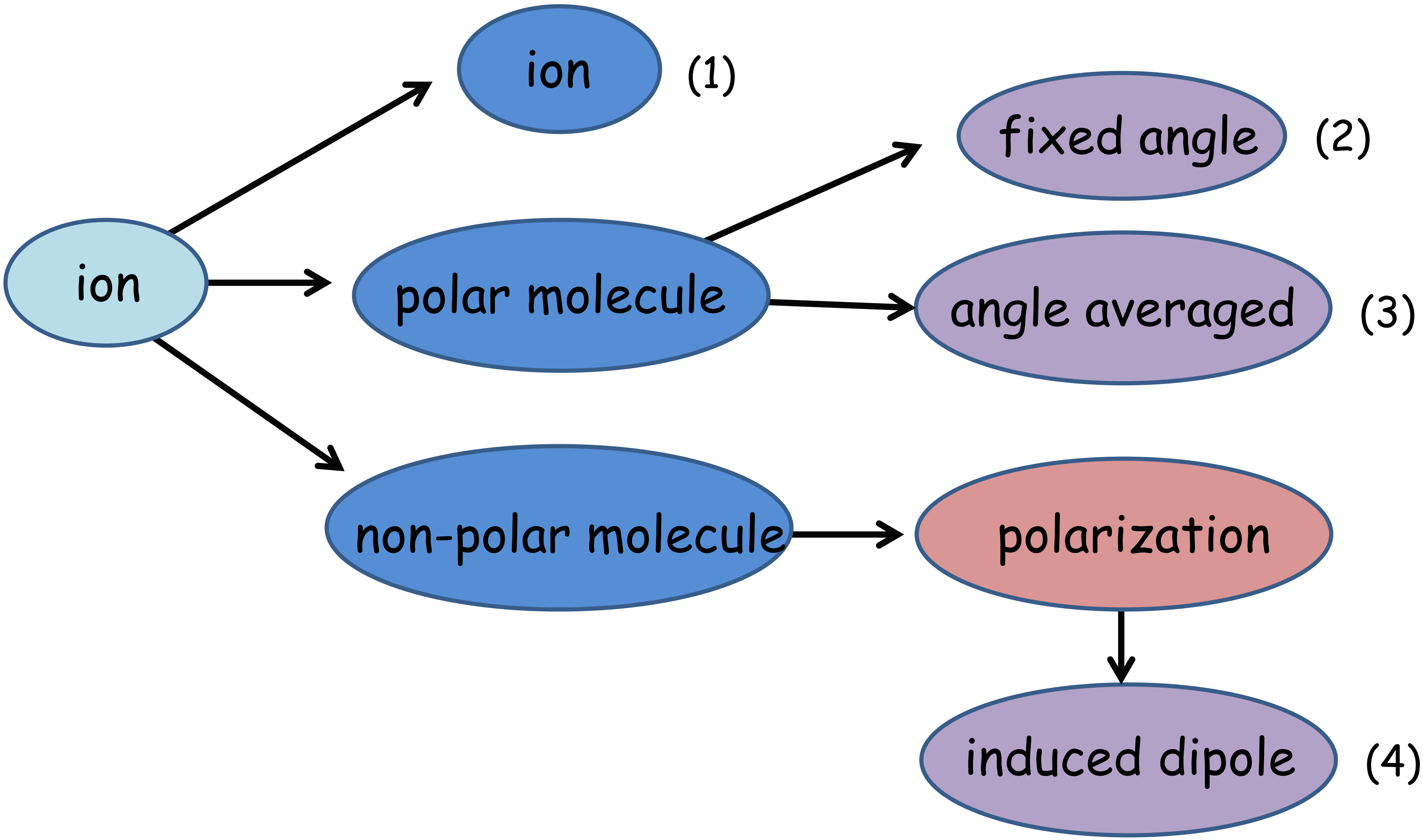
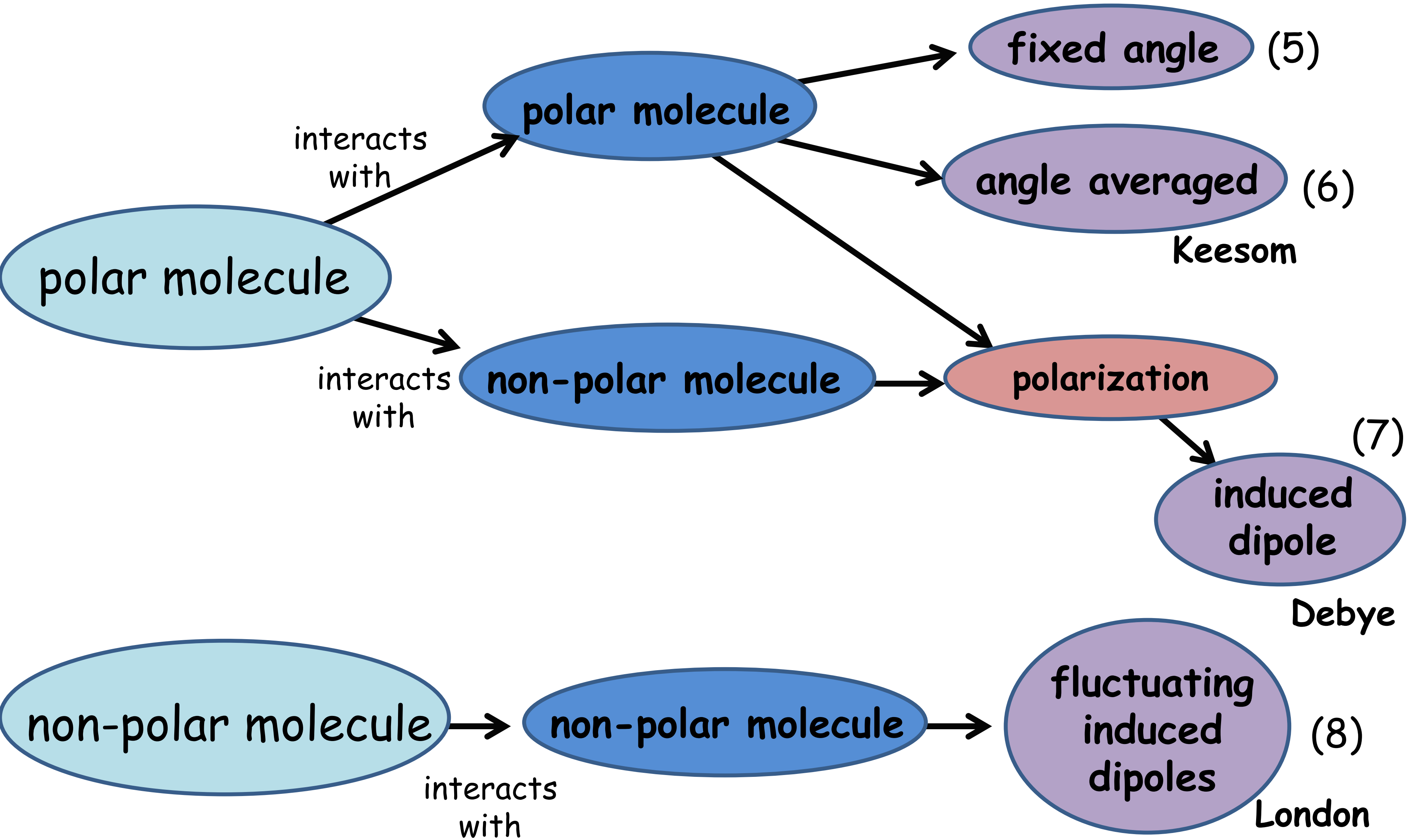
Last Lecture: Electrostatic Intermolecular Interactions



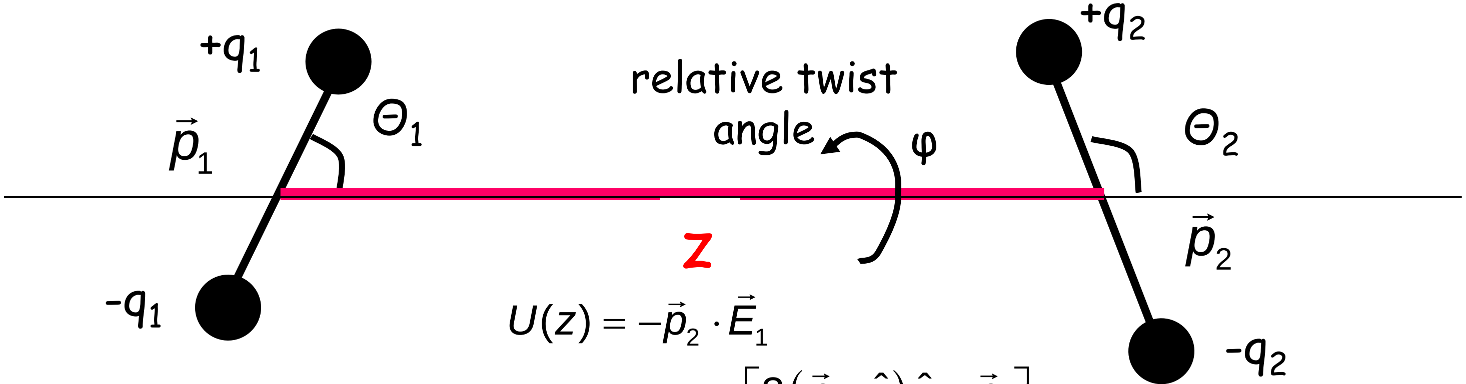
This Lecture: Electrostatic Intermolecular Interactions



Keesom:Debye:London≈100:10:1

5. Interaction between polar molecules: fixed orientation

already four parameters: separation distance (z) + 3 angles (Θ_1,Θ_2,φ)



$$U(z) = -\vec{p}_2 \cdot \vec{E}_1$$

$$\vec{E}_1 = \vec{E}_{1,dipole} = \frac{1}{4\pi\kappa\varepsilon_o}\frac{\left[3\left(\vec{p}_1\cdot\hat{z}\right)\hat{z} - \vec{p}_1\right]}{z^3}$$

$$\therefore U(z) = \frac{\vec{p}_1\cdot\vec{p}_2}{z^3} - 3\frac{\left(\vec{p}_1\cdot\hat{z}\right)\left(\vec{p}_2\cdot\hat{z}\right)}{z^3}$$

or

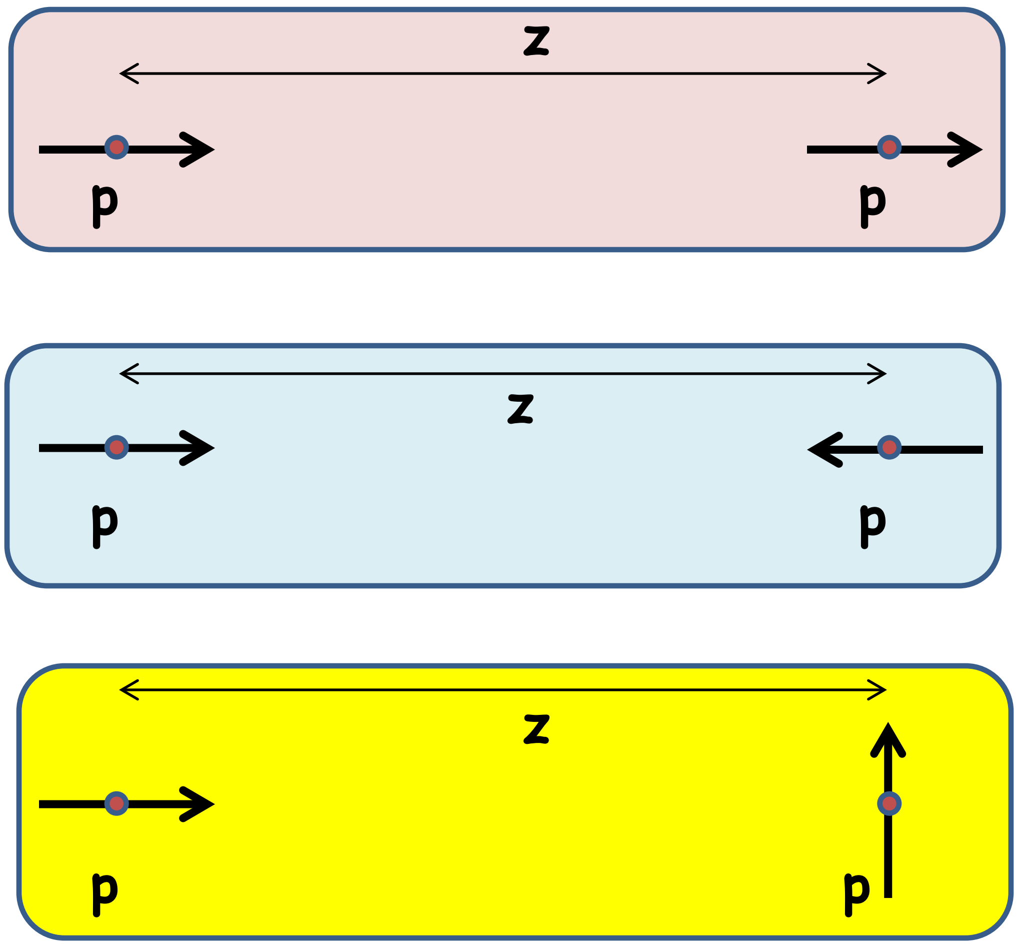
$$U(\theta_1,\theta_2,\varphi,z) = -\frac{p_1p_2\left[2\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\cos(\varphi)\right]}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}$$

$$= \frac{p_1p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}\cdot f\left(\theta_1,\theta_2,\varphi\right); \quad -2 \le f\left(\theta_1,\theta_2,\varphi\right) < 2$$

lowest energy when $\theta_1 = \theta_2 = 0 \Rightarrow U(0,0,\varphi,z) = -\frac{2p_1p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}$

Dipole-Dipole Interactions

The magnitude and sign of the dipole-dipole energy depends strongly on orientation

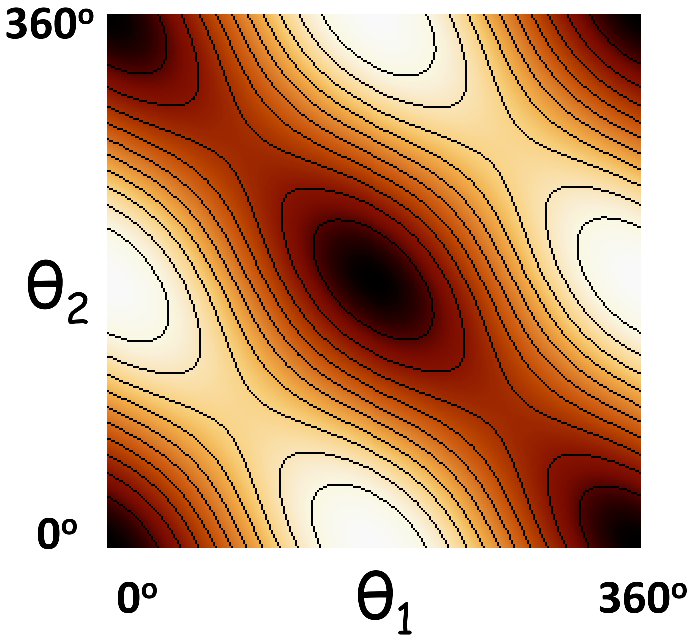


U(z) is minimum

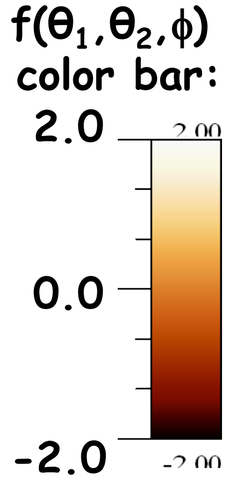
U(z) is maximum

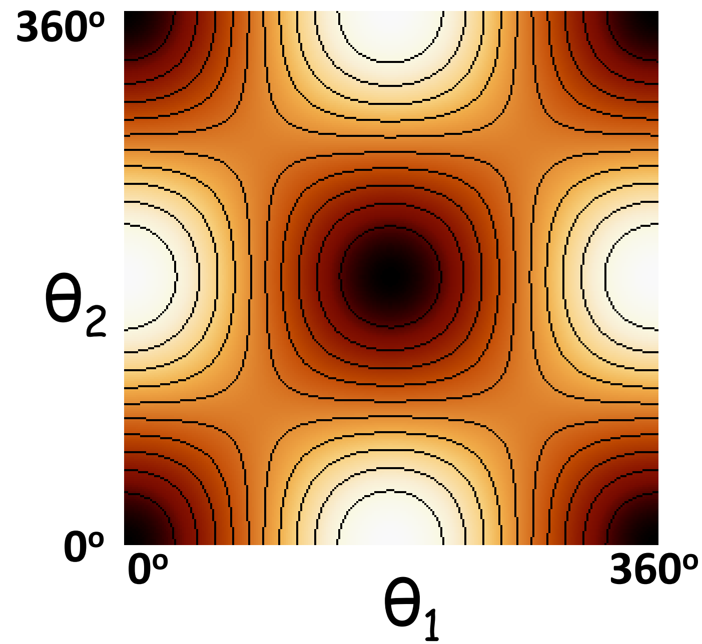
U(z) is zero

Representative plots of $f(\theta_1,\theta_2,\phi)$



ϕ=0; in plane configuration





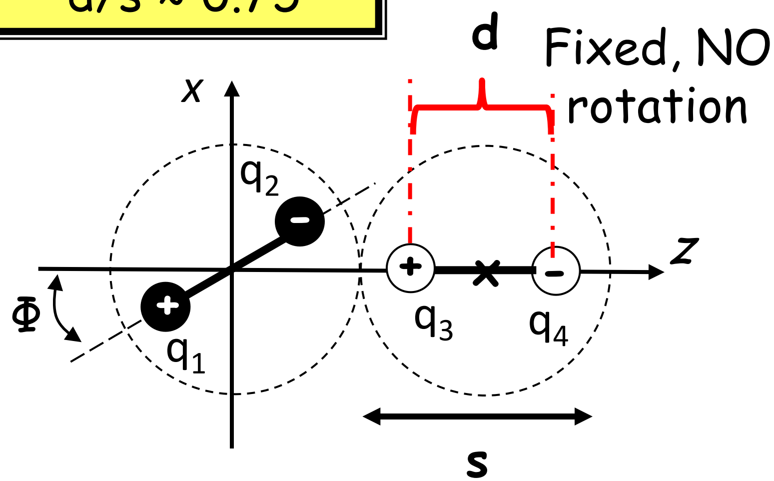
ϕ=90°; perpendicular configuration

$$U(\theta_1,\theta_2,\varphi,z)=\frac{p_1p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}f\left(\theta_1,\theta_2,\varphi\right)$$

$$f\left(\theta_1,\theta_2,\varphi\right)=-\left[2\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\cos(\varphi)\right]$$

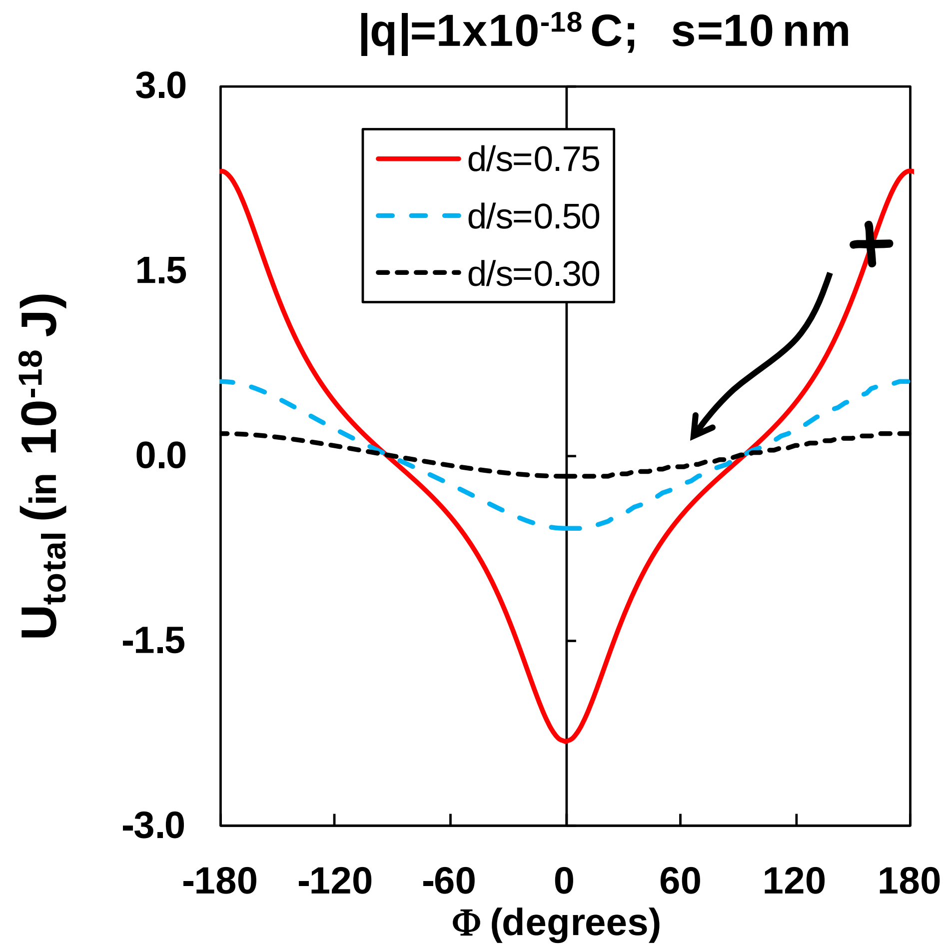
Interaction potential energy between in-plane (ϕ=0°) permanent dipoles

diagram for d/s ≈ 0.75



Symbol	Actual Charge
q_1	+ q
q_2	- q
q_3	+ q
q_4	- q

Note: $|p_1|=|p_2|$



6. Interaction between polar molecules: angle averaged

dipole-dipole interaction must be properly angle-averaged by allowing the orientation of dipoles p_1 and p_2 to fluctuate. Must calculate

$$\langle U(z)\rangle \equiv \left\langle U(z,\theta_1,\theta_2,\phi)\, e^{-U(z,\theta_1,\theta_2,\phi)/k_B T} \right\rangle$$

$$\textit{where} \quad U(z,\theta_1,\theta_2,\phi) = -\frac{p_1 p_2\left[2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\phi\right]}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}$$

here $\langle ....\rangle$ *represents the proper thermal average over all angles*

The complete derivation is given in Appendix A :

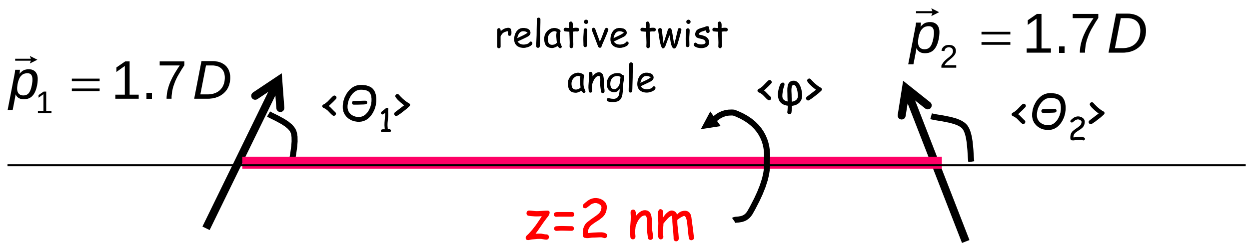
$$U_{Keesom}(z) = -\frac{2}{3}\frac{1}{k_B T}\left(\frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\right)^2\frac{1}{z^6} = -\frac{C_K}{z^6}$$

(also known as the Keesom interaction; W.H. Keesom, Physik. Z. **22**, 129 (1921) and **22**, 643 (1921))

The Keesom interaction between two ethanol molecules

from P1_Wk1_L3

Material	Chemical Formula	κ, dielectric constant[1]	p, dipole moment (in Debye)[2,3]	$\alpha_o/4\pi\varepsilon_o$ polarizability volume (in 10^{-30} m^3)
Ethanol	$(CH_3)CH_2OH$	24	1.7	5.1

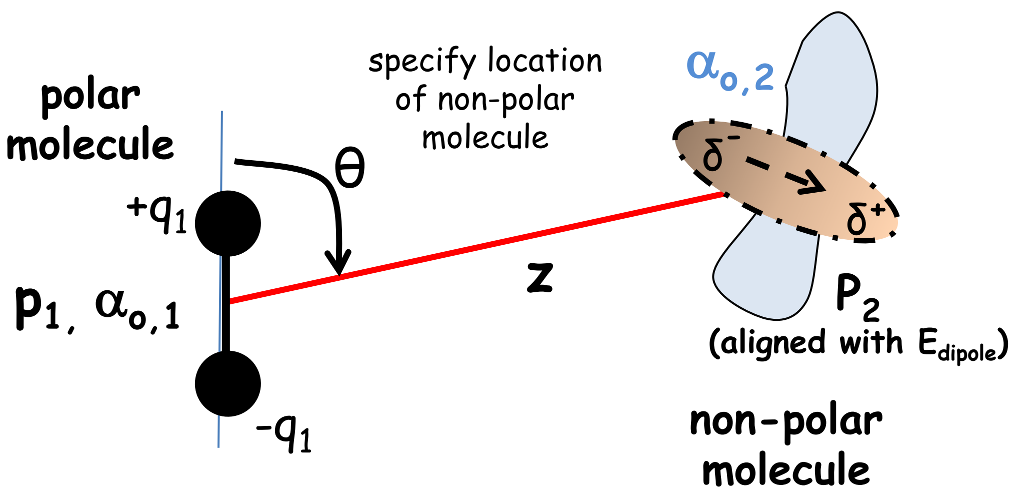


$$U_{Keesom}(z) = -\frac{2}{3}\frac{1}{k_B T}\left(\frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\right)^2 \frac{1}{z^6} = -\frac{2}{3}\frac{1}{(1.38\times10^{-23}\,J/K)(300K)}\left(\frac{\left((1.7\,D)\cdot\left(\frac{3.33\times10^{-30}\,Cm}{1D}\right)\right)^2}{4\pi\cdot(24)\cdot 8.85\times10^{-12}\,C^2/Nm^2}\right)^2 \frac{1}{(2\times10^{-9}\,m)^6}$$

$$= -\frac{2}{3}\left(\frac{1}{1.24\times 0^{-20}\,J}\right)\left(\frac{(5.66\times10^{-30})^2}{2.67\times10^{-9}}Nm^4\right)^2 \frac{1}{6.40\times10^{-53}\,m^6} = -(5.38\times10^{19})\cdot(1.44\times10^{-100})\cdot(1.56\times10^{52})\,J$$

$$= -12.08\times10^{-29}\,J\cdot\left(\frac{1eV}{1.6\times10^{-19}\,J}\right) = -7.55\times10^{-10}\,eV$$

7. Interaction potential energy between a polar and non-polar molecule



When a polar molecule is in close proximity to a non-polar molecule, the polarizing electric field is generated by the fixed dipole (p_1) of the polar molecule. This induces a dipole in the electron cloud of the non-polar molecule. The resulting interaction energy is called the Debye interaction energy.

$$\vec{E}_{dipole\ 1}(z,\theta) = \frac{|\vec{p}_1|}{4\pi\kappa\varepsilon_o}\frac{1}{z^3}\left[2\cos\theta\,\hat{z} + \sin\theta\,\hat{\theta}\right]$$

$$\left|\vec{E}_{dipole\ 1}(z,\theta)\right| = \frac{|\vec{p}_1|}{4\pi\kappa\varepsilon_o}\frac{1}{z^3}\left[4\cos^2\theta + \sin^2\theta\right]^{1/2}$$

$$= \frac{|\vec{p}_1|}{4\pi\kappa\varepsilon_o}\frac{1}{z^3}\left[3\cos^2\theta + 1\right]^{1/2}$$

$$U_{Debye}(z) = -2\cdot\frac{p_1^2\alpha_{o,2}}{(4\pi\kappa\varepsilon_0)^2}\frac{1}{z^6}$$

The induced dipole moment is a function of E

$$\vec{p}_{induced}(E) = \alpha_{o,2}\vec{E}$$

In this case, E is due to a dipole

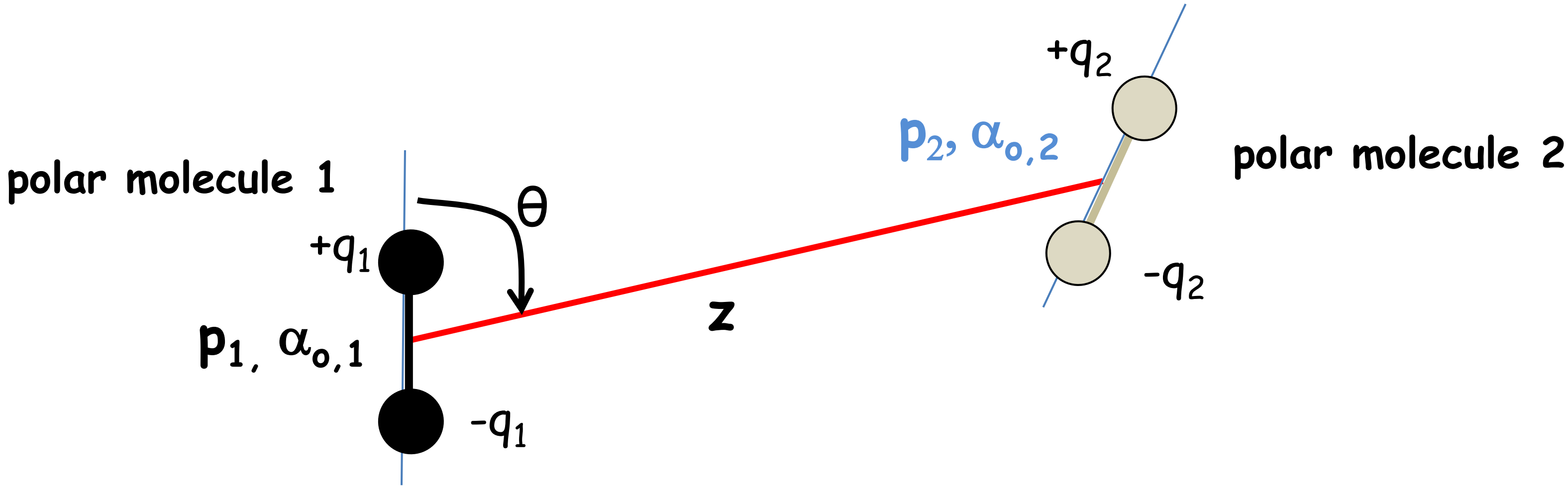
$$U_{Debye}(z,\theta) = -\vec{p}_{induced}(E)\cdot\vec{E} = \left(\alpha_{o,2}E_{dipole}^{2}\right)$$

$$= -\alpha_{o,2}\left(\frac{|\vec{p}_1|}{4\pi\kappa\varepsilon_o}\frac{\left[3\cos^2\theta+1\right]^{1/2}}{z^3}\right)^2 = -\frac{p_1^2\alpha_{o,2}}{(4\pi\kappa\varepsilon_o)^2}\frac{\left[3\cos^2\theta+1\right]}{z^6}$$

if an angle average over θ *is performed*, $\left\langle\cos^2\theta\right\rangle = \frac{1}{3}$ (See Appendix 1, P1_Wk1_L4)

$$U_{Debye}(z) = -2\cdot\frac{p_1^2\alpha_{o,2}}{(4\pi\kappa\varepsilon_o)^2}\frac{1}{z^6}$$

If the non-polar molecule is replaced by a polar one, then the second molecule could induce a dipole in the first:



The net effect is additive, so we must have

$$U_{Debye}(z) = -\frac{p_1^2\alpha_{o,2} + p_2^2\alpha_{o,1}}{(4\pi\kappa\varepsilon_0)^2}\frac{1}{z^6}$$

If the two polar molecules are identical, then

$$U_{Debye}(z) = -\frac{2p^2\alpha_o}{(4\pi\kappa\varepsilon_0)^2}\frac{1}{z^6}$$

Appendix A: Derivation of the angle-averaged Keesom interaction

The derivation of the angle-averaged Keesom interaction is usually not given in most textbooks. Rather, the final result is often written down after some statement like "After a lengthy calculation, we find........"

In what follows the derivation is given using the same logic followed when discussing the ion-diople interaction in the previous lecture (C1_Wk1_L3).

The starting point is the dipole-dipole interaction energy for two dipoles p_1 and p_2 separated by a distance z

$$U(z,\theta_1,\theta_2,\phi) = -\frac{p_1 p_2\left[2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\phi\right]}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}$$

At first, you might be tempted to perform a straightforward angle average of $U(z, \theta_1, \theta_2, \varphi)$ as follows:

$$U(z,\theta_1....) = -\frac{p_1 p_2 \left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi\right]}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}$$

$$Let \quad U(z,\theta_1...) = U_o(z)\left[2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi\right] = U_o(z) f(\Omega)$$

$$where \quad U_o(z) \equiv \frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3} \quad and \quad f(\Omega) = -2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi$$

$$\left\langle U(z)\right\rangle \equiv \frac{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} U(z,\theta_1,....)\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2} = U_o(z)\frac{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} f(\Omega)\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2} = U_o(z)\frac{I_A}{I_B}$$

$$I_A = \int_{\varphi=0}^{2\pi}\int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} \left[-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right]\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2 d\varphi$$

$$= -2\left[\int_{\varphi=0}^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} sin\theta_1 \cos\theta_1 d\theta_1\right]\left[\int_{\theta_2=0}^{\pi} sin\theta_2 \cos\theta_2 d\theta_2\right] + \left[\int_{\varphi=0}^{2\pi} \cos\varphi d\varphi\right]\left[\int_{\theta_1=0}^{\pi} \sin^2\theta_1 d\theta_1\right]\left[\int_{\theta_2=0}^{\pi} \sin^2\theta_2 d\theta_2\right] = 0$$

$$I_B = \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2 = 8\pi$$

$$\therefore \quad \left\langle U(z)\right\rangle = U_o(z)\cdot\frac{I_A}{I_B} = 0$$

The result is zero because the angle average is performed for a "free" dipole in which all angles are equally likely. Instead, a weighted angle average must be calculated that preferentially weights those angles having lowest energy. This involves a correct treatment of the Boltzmann weighting factor and requires some understanding of statistical thermodynamics.

Weighting the angles using the Boltzmann factor gives

$$\left\langle U(z)\right\rangle \equiv \left\langle U(z,\theta_1,...)\, e^{-U(z,\theta_1,...)/k_BT}\right\rangle \equiv \frac{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} U(z,\theta_1,....)\; e^{-U(z,\theta_1,...)/k_BT}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} e^{-U(z,\theta_1,...)/k_BT}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}$$

$$let \quad \beta = -\frac{U_o(z)}{k_BT}, \quad then$$

$$\left\langle U(z,\theta_1,....)e^{-U(z,\theta_1,....)/k_BT}\right\rangle = U_o(z)\frac{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} f(\Omega)\; e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}{\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2}$$

Evaluating such a quantity is simplified by recognizing that the numerator and denominator are related by a derivative. Using the well known properties of the natural logarithm:

$$\left\langle U(z,\theta_1,....)e^{-U(z,\theta_1,....)/k_BT}\right\rangle = U_o(z)\frac{d}{d\beta}\ln\left(\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2\right)$$

$$since \quad \frac{d}{d\beta}\ln\left(\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2\right)$$

$$= \frac{1}{\left(\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2\right)} \times \left(\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} f(\Omega)e^{\beta f(\Omega)}\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2\right)$$

This means we must evaluate the integral $\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)} \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$

if $\beta f(\Omega) << 1$, *we can write*

$$\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)} \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2 \simeq \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[1 + \beta f(\Omega) + \frac{\beta^2}{2} f^2(\Omega) + \ldots\right] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} [1] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$+ \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} [\beta f(\Omega)] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$+ \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[\frac{\beta^2}{2} f^2(\Omega)\right] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2 + \ldots$$

$$= I_1 + I_2 + I_3 + \ldots$$

$$I_1 = I_B = 8\pi \qquad I_2 = 0$$

$$I_3 = \frac{\beta^2}{2} \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[-2\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi\right]^2 \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= \frac{\beta^2}{2} \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[4\cos^2\theta_1 \cos^2\theta_2\right] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$+ \frac{\beta^2}{2} \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[-4\cos\theta_1 \cos\theta_2 \sin\theta_1 \sin\theta_2 \cos\phi\right] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$+ \frac{\beta^2}{2} \int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \left[\sin^2\theta_1 \sin^2\theta_2 \cos^2\phi\right] \sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= I_a + I_b + I_c$$

$$I_a = 4\cdot\frac{\beta^2}{2}\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi}\left[\cos^2\theta_1\cos^2\theta_2\right]\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= 4\cdot\frac{\beta^2}{2}[2\pi]\left[-\frac{1}{3}\cos^3\theta_1\right]\Bigg|_{\theta_1=0}^{\theta_1=\pi}\left[-\frac{1}{3}\cos^3\theta_2\right]\Bigg|_{\theta_2=0}^{\theta_2=\pi} = 4\cdot\frac{\beta^2}{2}[2\pi]\cdot\left[-\frac{1}{3}\cdot\left[(-1)^3-(1)^3\right]\right]^2 = 4\pi\beta^2\cdot\frac{4}{9} = \frac{16}{9}\pi\beta^2$$

$$I_b = -4\cdot\frac{\beta^2}{2}\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi}\left[\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\cos\phi\right]\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= -4\cdot\frac{\beta^2}{2}\int_0^{2\pi}\cos\varphi d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi}\left[\cos\theta_1\sin^2\theta_1\cos\theta_2\sin^2\theta_2\right]d\theta_1\, d\theta_2$$

$$= -4\cdot\frac{\beta^2}{2}[\sin\varphi]\Big|_{\varphi=0}^{\varphi=2\pi}\left[\frac{1}{3}\sin^3\theta_1\right]\Bigg|_{\theta_1=0}^{\theta_1=\pi}\left[\frac{1}{3}\sin^3\theta_2\right]\Bigg|_{\theta_2=0}^{\theta_2=\pi} = 0$$

$$I_c = \frac{\beta^2}{2}\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi}\left[\sin^2\theta_1\sin^2\theta_2\cos^2\phi\right]\sin\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2$$

$$= \frac{\beta^2}{2}\int_0^{2\pi}\cos^2\phi d\varphi \int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi}\left[\sin^3\theta_1\sin^3\theta_2\right]d\theta_1\, d\theta_2$$

$$= \frac{\beta^2}{2}\left[\frac{1}{2}(\varphi+\sin\varphi\cos\varphi)\right]\Bigg|_{\varphi=0}^{\varphi=2\pi}\left[\frac{1}{12}(\cos 3\theta_1 - 9\cos\theta_1)\right]\Bigg|_{\theta_1=0}^{\theta_1=\pi}\left[\frac{1}{12}(\cos 3\theta_2 - 9\cos\theta_2)\right]\Bigg|_{\theta_2=0}^{\theta_2=\pi}$$

$$= \frac{\beta^2}{2}[\pi]\cdot\left[\frac{1}{12}\left[-1-9(-1)-(1-9(1))\right]\right]^2 = \frac{1}{2}\pi\beta^2\left[\frac{8}{12}+\frac{8}{12}\right]^2 = \frac{1}{2}\pi\beta^2\left[\frac{4}{3}\right]^2 = \frac{8}{9}\pi\beta^2$$

$$I_3 = I_a + I_b + I_c = \frac{16}{9}\pi\beta^2 + 0 + \frac{8}{9}\pi\beta^2 = \frac{24}{9}\pi\beta^2 = \frac{8}{3}\pi\beta^2$$

$$So \quad I_1 + I_2 + I_3 + ... = 8\pi \cdot \left(1 + \frac{\beta^2}{3} + ...\right)$$

$$\left\langle U(z) \right\rangle \equiv \left\langle U(z,\theta) e^{-U(z,\theta)/k_B T} \right\rangle = U_o(z) \frac{d}{d\beta} \ln\left(\int_0^{2\pi} d\varphi \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} e^{\beta f(\Omega)} \sin\theta_1 \, d\theta_1 \sin\theta_2 \, d\theta_2 \right)$$

$$= U_o(z) \frac{d}{d\beta} \ln\left(I_1 + I_2 + I_3 + ...\right)$$

$$= U_o(z) \frac{d}{d\beta} \ln\left(8\pi \left(1 + \frac{\beta^2}{3} + ...\right)\right) = U_o(z) \frac{d}{d\beta}\left[\ln\left(8\pi\right) + \ln\left(1 + \frac{\beta^2}{3} + ...\right)\right]$$

$$= 0 + U_o(z) \frac{1}{1 + \frac{\beta^2}{3}} \cdot \frac{2\beta}{3}$$

$$\approx U_o(z) \frac{2\beta}{3}$$

now since $\beta = -\dfrac{U_o(z)}{k_B T}$ *and* $U_o(z) \equiv \dfrac{p_1 p_2}{4\pi\kappa\varepsilon_0}\dfrac{1}{z^3}$,

we finally obtain

$$\langle U(z)\rangle \equiv \left\langle U(z,\theta_1,\ldots)e^{-U(z,\theta_1,\ldots)/k_B T}\right\rangle = U_o(z)\frac{2\beta}{3}$$

$$= \left(\frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}\right)\cdot\frac{2}{3}\cdot\left(-\frac{1}{k_B T}\cdot\frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\frac{1}{z^3}\right)$$

$$= -\frac{2}{3k_B T}\left(\frac{p_1 p_2}{4\pi\kappa\varepsilon_0}\right)^2\frac{1}{z^6}$$