Lecture: P1_Wk4_L1

Cantilever Mechanics – The Force Sensor

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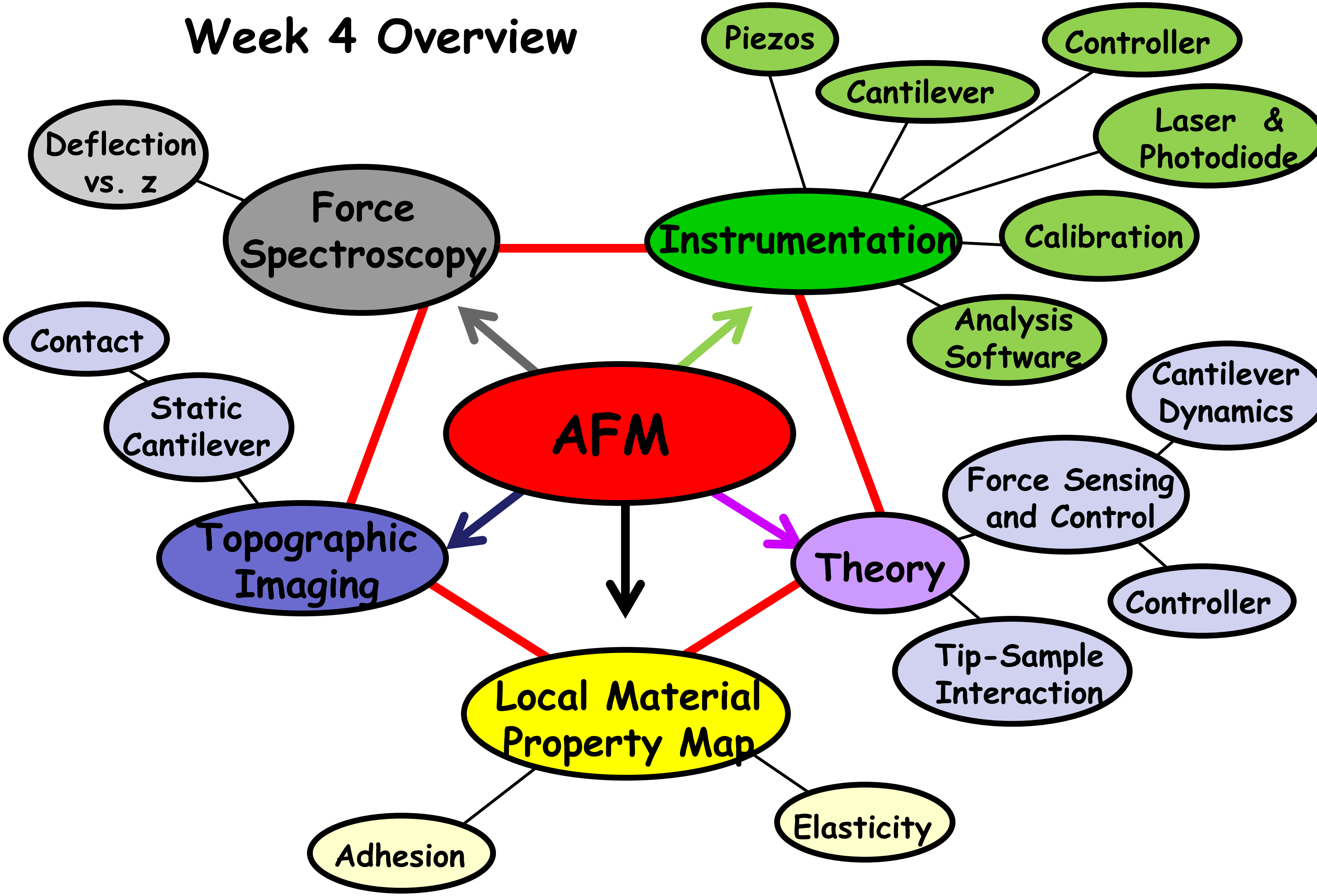
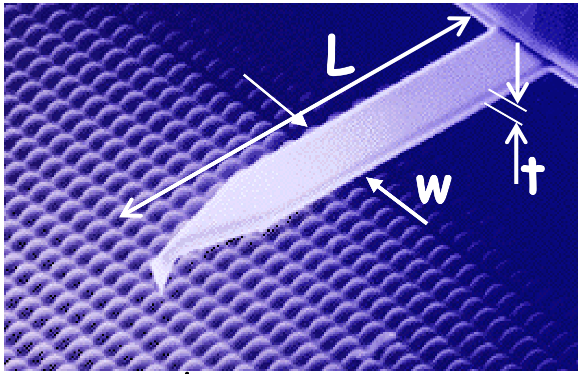
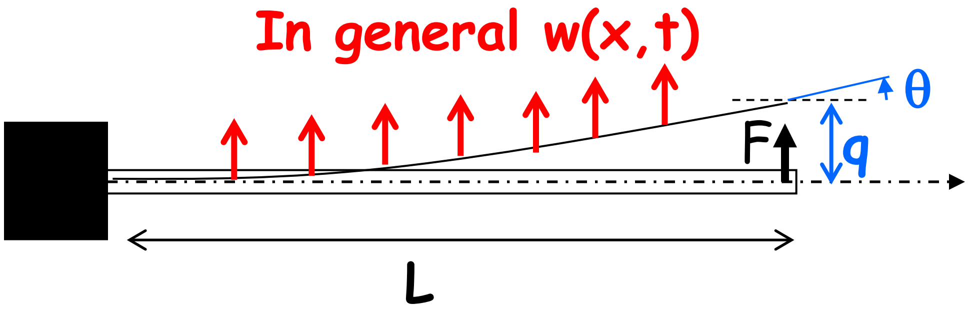
Week 4 Overview



The Loaded Microcantilever



www.olympus.co.jp



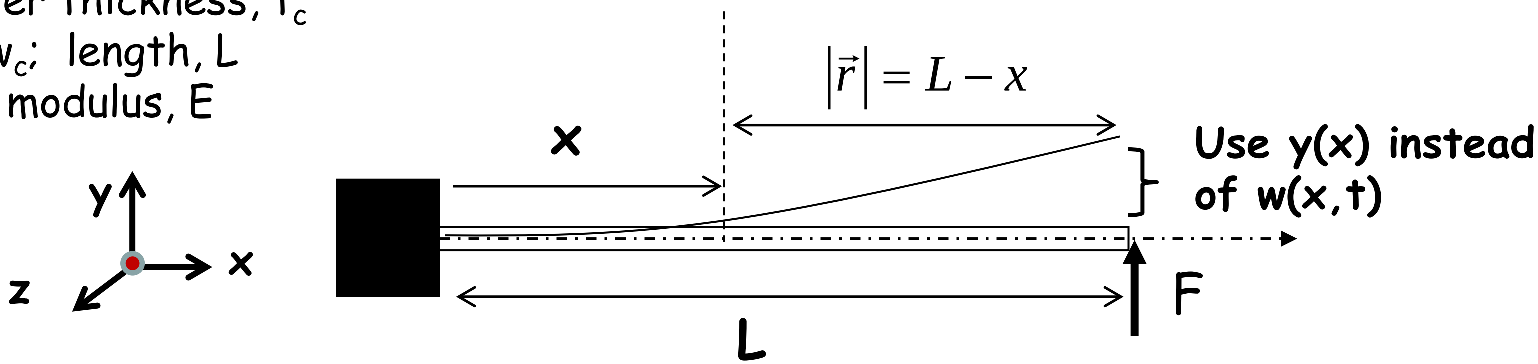
Notation: cantilever thickness, t_c
width, w_c

q is the deflection of the tip at x=L, w(L,t).

- If the cantilever is made from a material with a Young's modulus E, what is the deflection shape w(x,t), the bending stiffness k (spring constant), and the slope at x=L when a point force F is applied?
- Is deflection proportional to F?

The problem: What is y(x)?

cantilever thickness, t_c
width, w_c; length, L
Young's modulus, E



Torque:

$$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}| sin\theta(\hat{z})$$

$$if\ \vec{r} \perp \vec{F}, then\ |\vec{\tau}| = |\vec{r}||\vec{F}| \quad (Bernoulli - Euler\ assumption)$$

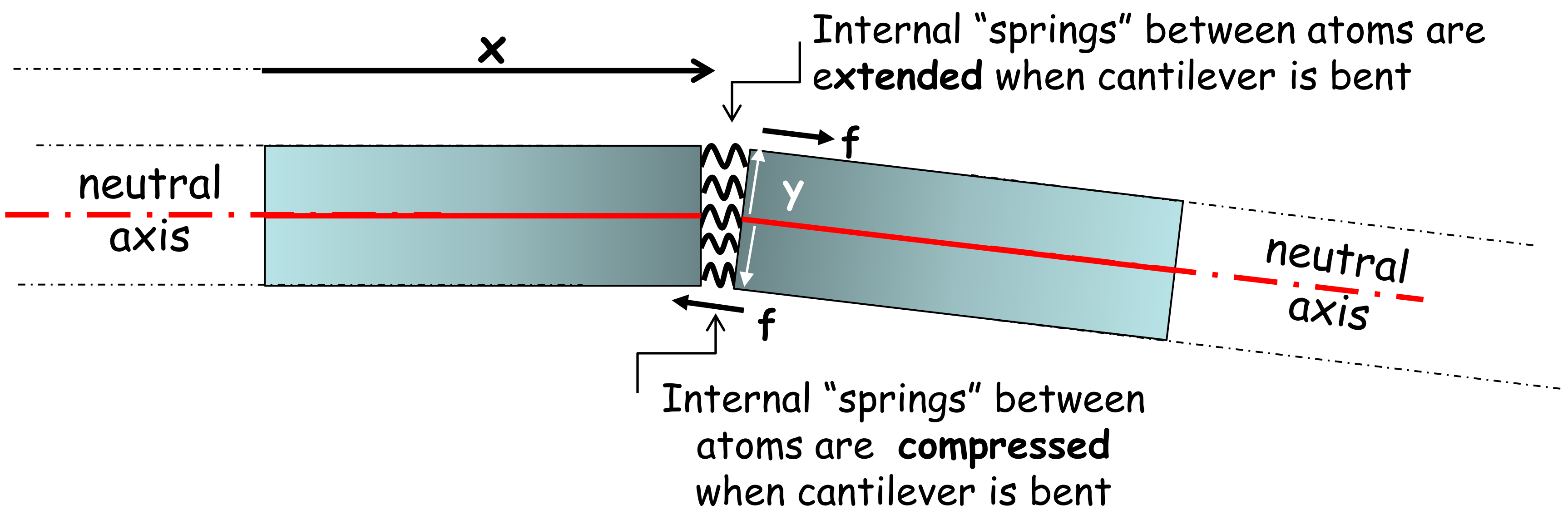
If torque refers to bending of a rigid object, it is often called a moment M rather than a torque τ. This distinction makes it clear we are discussing <u>bending</u> rather than <u>twisting</u>.

The magnitude of the bending moment at point x is

$$|\vec{M}| = F(L - x)$$

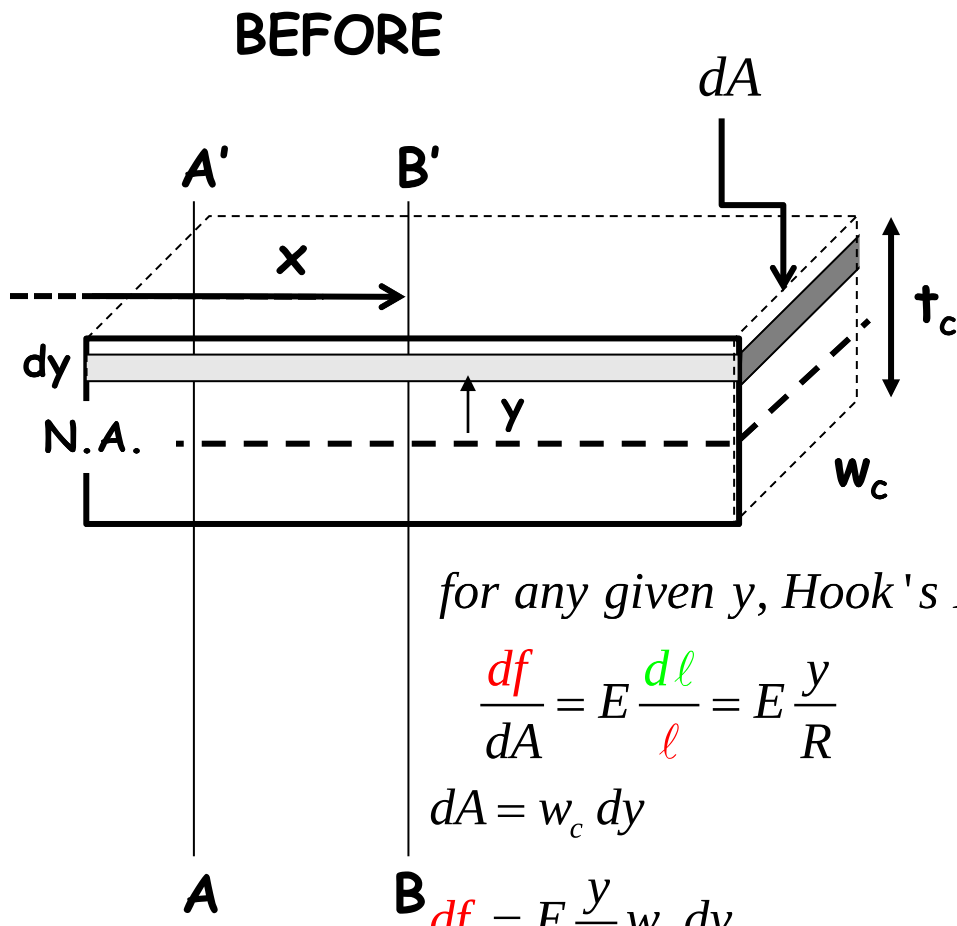
In equilibrium, all **internal** moments developed at x by the cantilever due to bending must balance the applied moment M at point x.

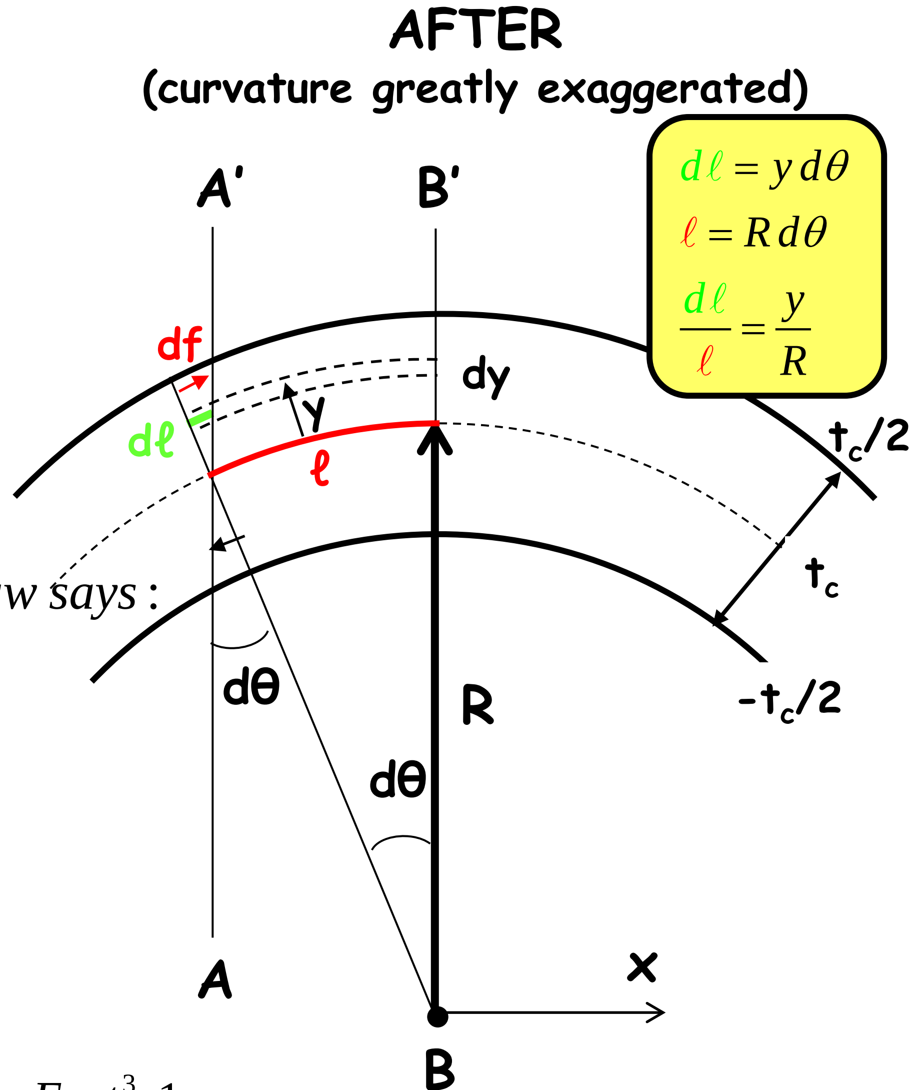
Internal moments develop because of internal stresses



In equilibrium, "**torques balance torques**".

This means each internal "spring" applies a moment around the neutral axis that tends to restore the cantilever to its original position. The horizontal "spring" forces (f) are thus translated into a vertical force that tends to restore the cantilever to its unbent shape.





for any given y, Hook's Law says:

$$\frac{df}{dA} = E\frac{d\ell}{\ell} = E\frac{y}{R}$$

$$dA = w_c\, dy$$

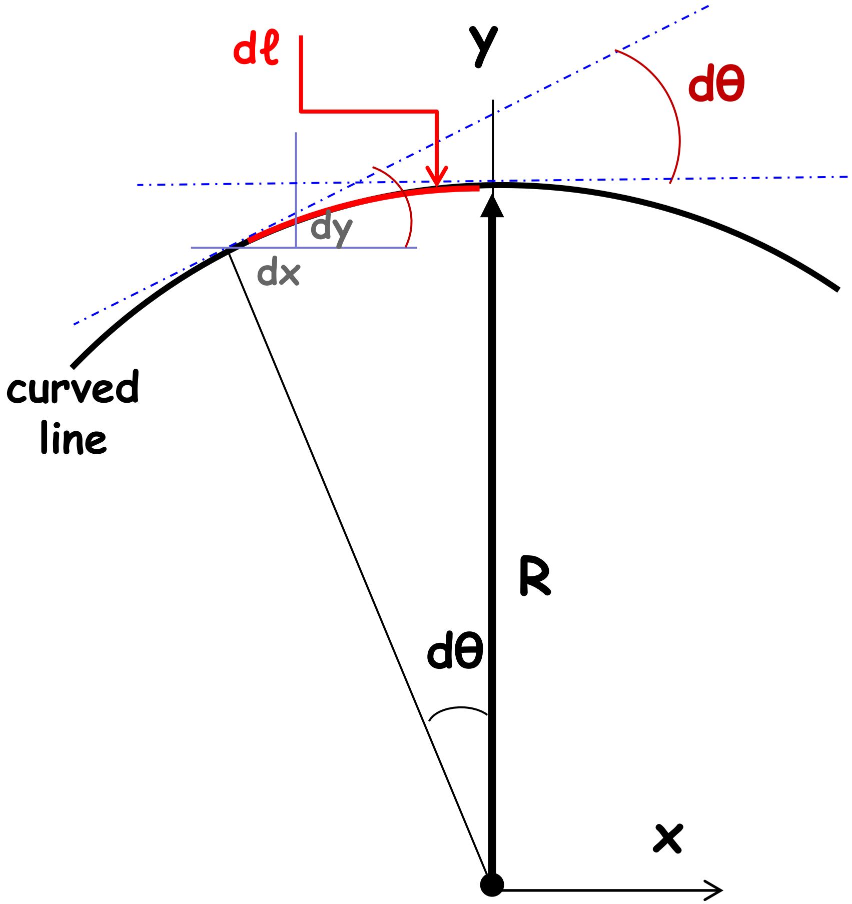
$$df = E\frac{y}{R} w_c\, dy$$

applied moment = internal moment:

$$M_{applied} = F(L-x) = \int_{-t_c/2}^{+t_c/2} df \cdot y = \int_{-t_c/2}^{+t_c/2} E\frac{y^2}{R} w_c\, dy = \frac{E w_c t_c^3}{12}\frac{1}{R}$$

$$\therefore \quad \text{curvature of beam at } x \equiv \kappa = \frac{1}{R} = \frac{12F(L-x)}{E w_c t_c^3}$$

Review - defining the curvature of a line



$$\kappa \equiv \frac{\text{change in angle of tangent line}}{\text{change in arc length}} = \frac{d\theta}{d\ell} = \frac{d\theta}{dx}\frac{dx}{d\ell}$$

$$\tan\theta = \frac{dy}{dx} \quad \Rightarrow \quad \theta = \operatorname{atan}\left(\frac{dy}{dx}\right)$$

$$\frac{d\theta}{dx} = \frac{d}{dx}\left[\operatorname{atan}\left(\frac{dy}{dx}\right)\right] = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}$$

$$(d\ell)^2 = (dx)^2 + (dy)^2$$

$$d\ell = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{dx}{d\ell} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\therefore \quad \kappa \equiv \frac{d\theta}{d\ell} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{1}{R}$$

If curved line is a circle:

$$\kappa \equiv \frac{\text{change in angle of tangent line}}{\text{change in arc length}}$$

$$= \frac{2\pi}{2\pi R} = \frac{1}{R} \left(\text{in } {}^{rad}\!/_{m}\right)$$

Deflection (y) vs. x:

$$\text{if the curvature is small, then } \frac{dy}{dx} \ll 1$$

$$\therefore \quad \frac{1}{R} \simeq \frac{d^2y}{dx^2} = \frac{12F(L-x)}{Ew_c t_c^3}$$

$$y(x) = \iint d^2y = \iint \frac{12F(L-x)}{Ew_c t_c^3} dx^2$$

$$= \frac{12F}{Ew_c t_c^3}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

$$\text{when } x = L$$

$$q \equiv y(L) = \frac{4L^3}{Ew_c t_c^3} F$$

$F = kq \Rightarrow$ $k = \dfrac{Ew_c t_c^3}{4L^3}$ **spring constant**

Slope (Θ) vs. x:

$$\Theta(x) = \frac{dy}{dx} = \frac{12F}{Ew_c t_c^3}\left(Lx - \frac{x^2}{2}\right)$$

$$\text{when } x = L$$

$$\Theta(L) = \frac{6L^2}{Ew_c t_c^3} F$$

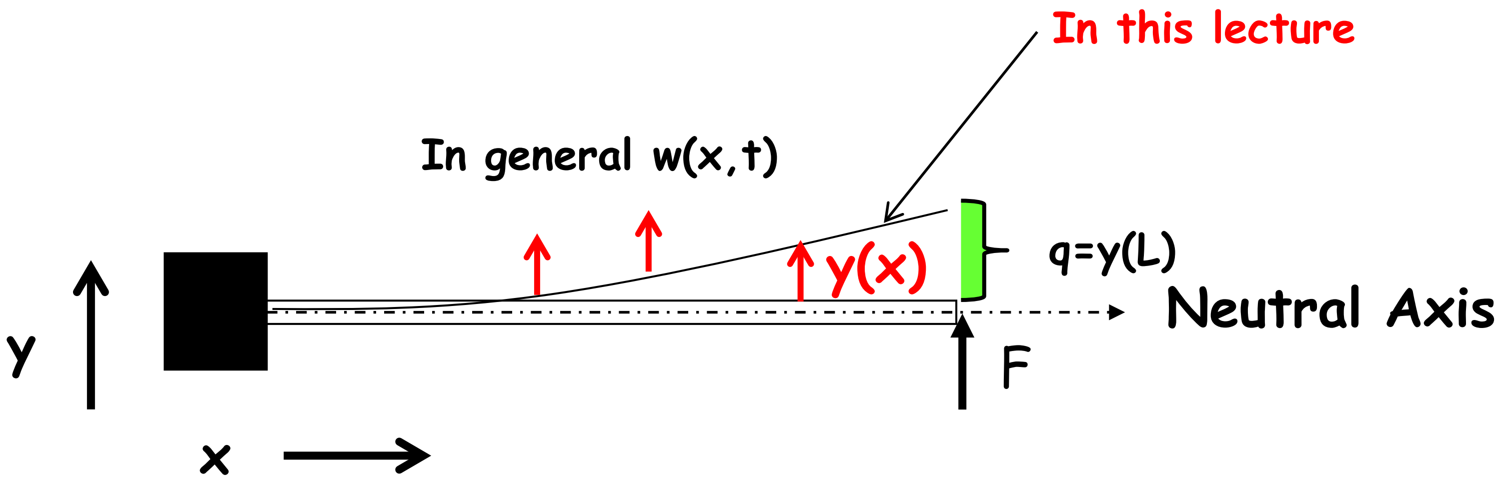
$$F = c\,\Theta(L) \Rightarrow c = \frac{2}{3} kL$$

$$\Theta(L) = \frac{3}{2}\frac{q}{L}$$

Standard results in terms of <u>area</u> moment, I [units: m^4]:

$$I \equiv \int_{-t_c/2}^{+t_c/2} w_c\, y^2 dy = \frac{1}{12} w_c t_c^3; \quad q \equiv y(L) = \frac{L^3}{3EI} F; \quad k = \frac{3EI}{L^3}; \quad \Theta(L) = \frac{L^2}{2EI} F$$

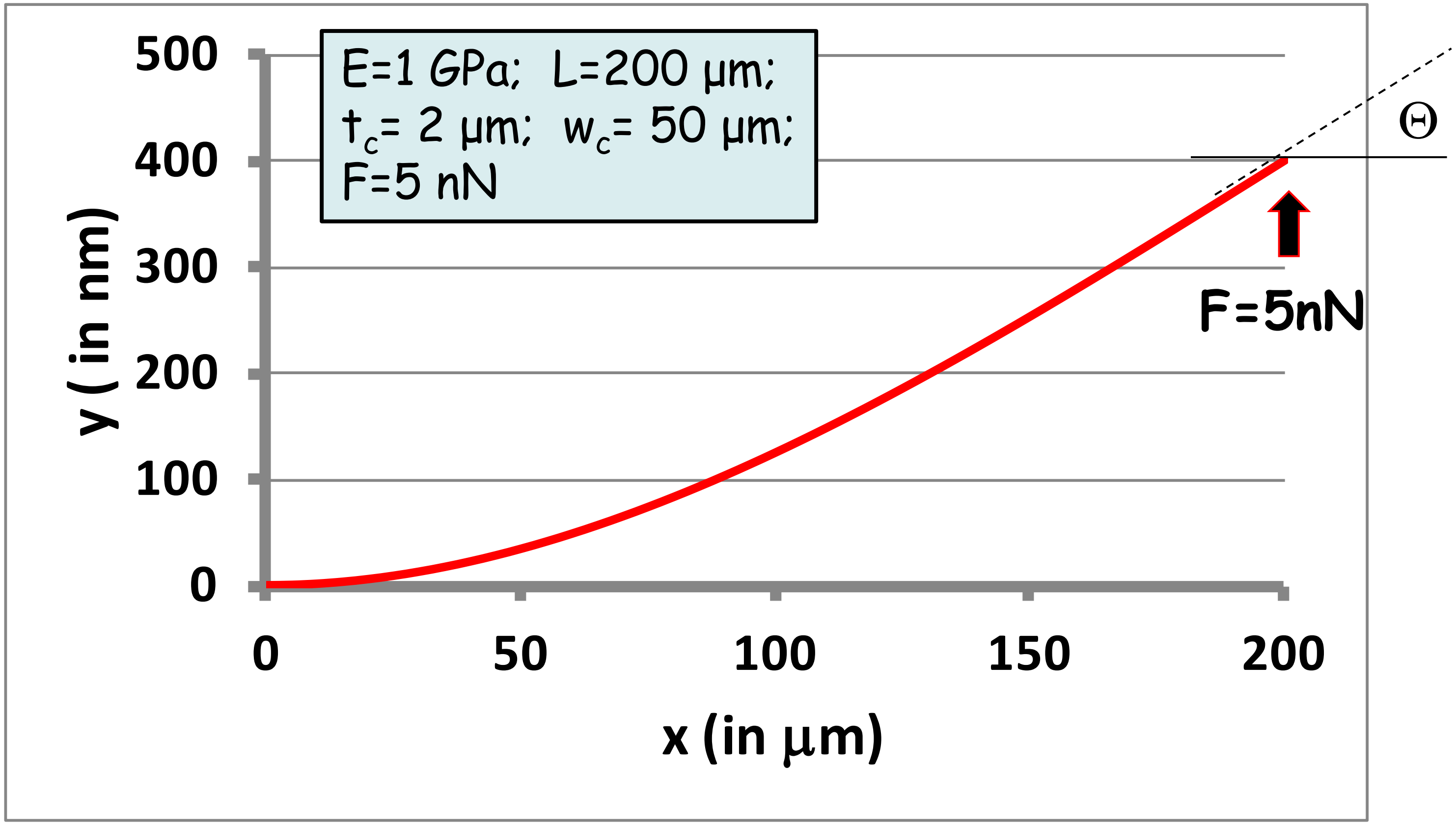
Notation



Converting to cantilever coordinates:

$y(x) = w(x)$
$y(L) = w(L) \equiv q$

Profile of cantilever



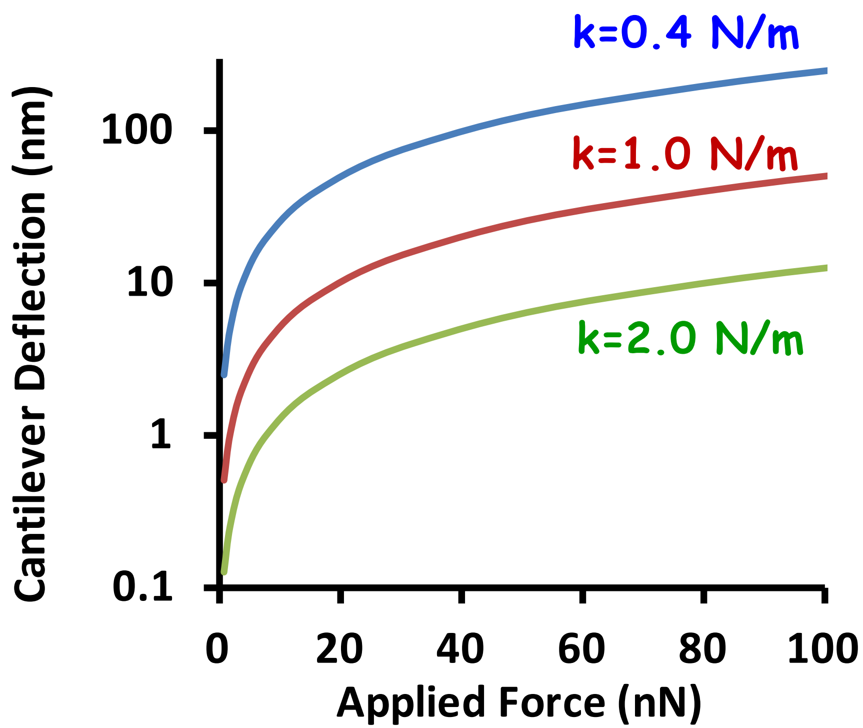
Linear and Angular Deflection Plots

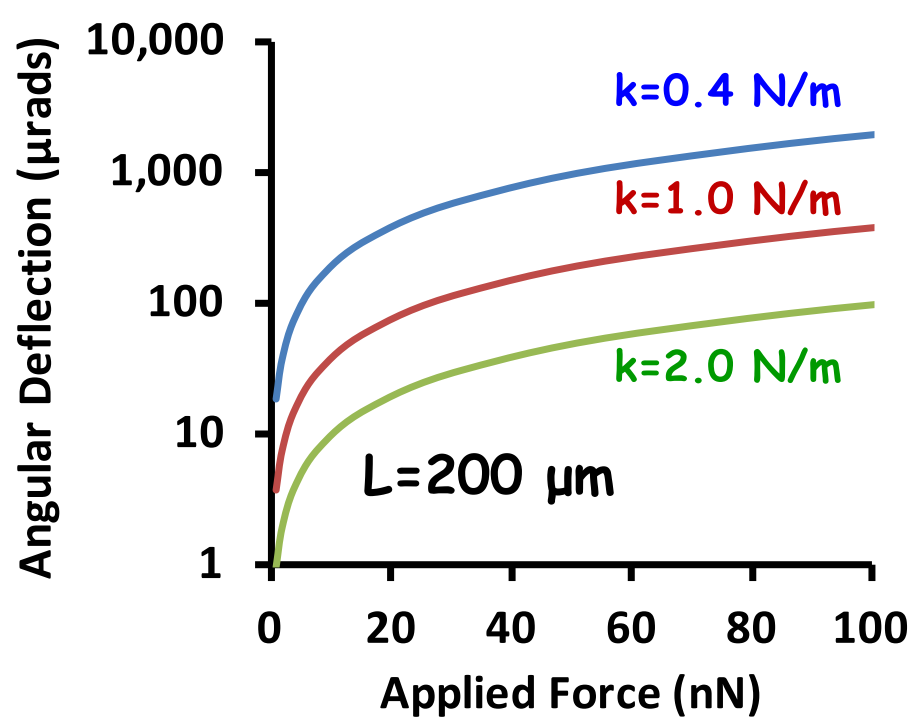
$$q \equiv y(L) = \frac{4L^3}{Ew_c t_c^3} F$$

$$F = kq \Rightarrow k = \frac{Ew_c t_c^3}{4L^3}$$

$$\Theta(L) = \frac{6L^2}{Ew_c t_c^3} F$$

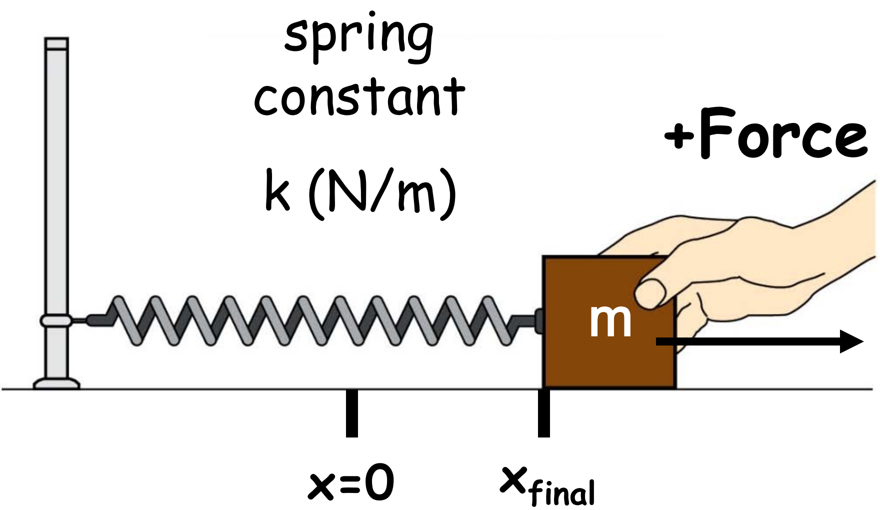
$$F = c\,\Theta(L) \Rightarrow c = \frac{2}{3} kL$$



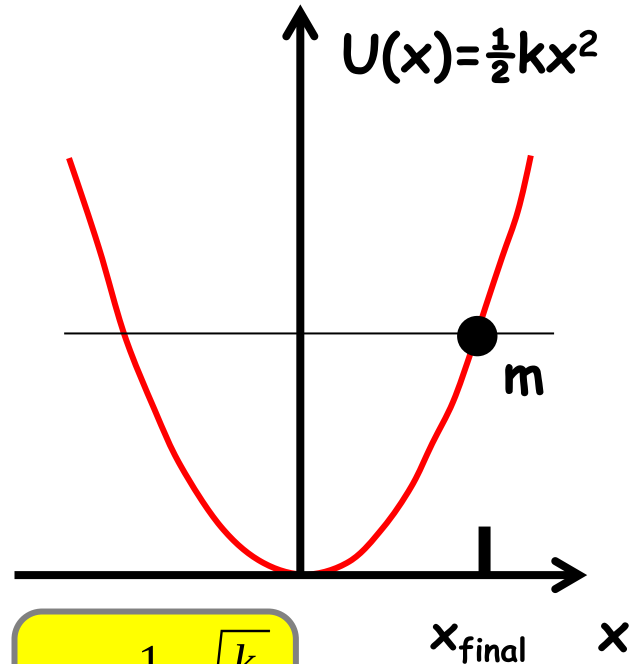


Implications – Resonant Frequency

Work Done = Elastic Energy Stored



$$F_{restore} = -kx_{final}$$

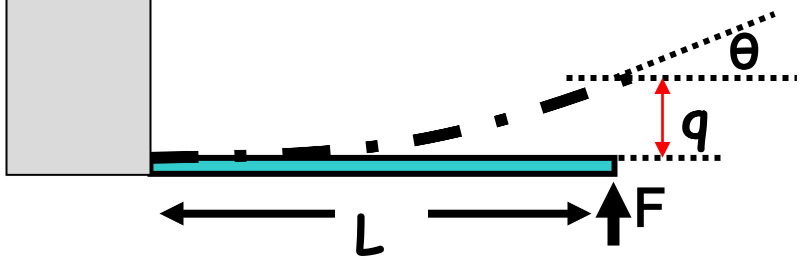


$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Caution: when comparing to cantilever, note that each part of the cantilever is displaced by a different amount. Need to introduce an **effective mass** to describe this effect.

Summary

Static model



$$q = \frac{FL^3}{3EI} \qquad \theta = \frac{FL^2}{2EI}$$

$$k = \frac{3EI}{L^3} = \frac{Ew_c t_c^3}{4L^3}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{eff}}} = \frac{1}{2\pi}\sqrt{\frac{Ew_c t_c^3}{4L^3 m_{eff}}}$$

$$\text{where } m_{eff} = 0.24 m_{cant}$$

$$\text{and } m_{cant} = \rho L w_c t_c$$

Dynamic model

(Euler-Bernoulli Equation)

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho w_c t_c \frac{\partial^2 w(x,t)}{\partial t^2} = 0$$

$$f_i = \frac{1}{2\pi}\beta_i^2\sqrt{\frac{EI}{\rho A}} = \frac{1}{2\pi}\beta_i^2\sqrt{\frac{E\,t_c^2}{12\rho}}$$

First eigenmode (i=1)

$$f_1 = \frac{1}{2\pi}\left(\frac{\beta_i}{L}\right)^2\sqrt{\frac{EI}{\rho A}} = \frac{1}{2\pi}\frac{(1.876)^2}{\sqrt{3}}\sqrt{\frac{Ew_c t_c^3}{4L^3 m_c}}$$

Will be discussed in Part II of this course

Up Next: Tip Approach to the Substrate

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