LECTURE #2

THE STATISTICAL INTERPRETATION OF $\Psi(x,t)$ IMPLIES:

$$\boxed{\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1}$$ (A.K.A, NORMALIZATION CONDITION)

(i.e, THE PARTICLE MUST BE SOMEWHERE)

NOTE: THE SCHRÖDINGER EQUATION IS A LINEAR DIFFERENTIAL EQUATION $\Rightarrow$ IF $\Psi(x,t)$ IS A SOLUTION, THEN $A\Psi(x,t)$ IS ALSO A SOLUTION (WHERE A IS ANY COMPLEX CONSTANT)

$\therefore$ AFTER $\Psi(x,t)$ IS FOUND FROM THE SCHRÖDINGER EQUATION, A IS CALCULATED USING THE NORMALIZATION CONDITION

EVIDENTLY, ANY SOLUTION TO THE SCH. EQ. THAT FAILS THE NORMALIZATION CONDITION IS NON-PHYSICAL AND MUST BE REJECTED.

$\Rightarrow$ $\Psi(x,t)$ MUST GO TO ZERO FASTER THAN $\frac{1}{\sqrt{|x|}}$ AS $|x| \to \infty$

IF Ψ HAS BEEN NORMALIZED AT $t=0$, WILL IT REMAIN NORMALIZED FOR FUTURE VALUES OF t?

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2 dx$$

NOTE: $\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$

SCH. EQ.: $\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$

COMPLEX CONJUGATE OF SCH. EQ.: $\frac{\partial \Psi^*}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V \Psi^*$

$$\therefore \frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]$$

$$\therefore \frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{\infty}$$

SINCE Ψ (AND Ψ^*) $\to 0$ AS $|x| \to \infty$,

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 0$$

$\Rightarrow$ **IF Ψ IS NORMALIZED AT $t=0$, IT REMAINS NORMALIZED FOR ALL FUTURE VALUES OF t.**

IN QUANTUM MECHANICS, THE EXPECTATION VALUE (i.e., AVERAGE VALUE --- WHICH IS DIFFERENT THAN THE MOST PROBABLE VALUE) OF x IS:

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, |\Psi(x,t)|^2 dx$$

SINCE Ψ IS TIME DEPENDENT, $\langle x \rangle$ WILL CHANGE WITH TIME.

NOTE SUPPRESSION OF LIMITS OF INTEGRATION

$$\therefore \frac{d\langle x \rangle}{dt} = \int x \frac{\partial}{\partial t}|\Psi|^2 dx = \frac{i\hbar}{2m}\int x \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx$$

INTEGRATE BY PARTS: $u = x$ $\quad dv = \frac{\partial}{\partial x}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right) dx$

$du = dx$ $\quad v = \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi$

$$\therefore \frac{d\langle x\rangle}{dt} = \frac{-i\hbar}{2m}\int\Big(\Psi^*\frac{\partial\Psi}{\partial x} - \underbrace{\frac{\partial\Psi^*}{\partial x}\Psi}\Big)dx$$

INTEGRATE BY PARTS:

$u = \Psi \qquad dv = -\frac{\partial\Psi^*}{\partial x}dx$

$du = \frac{\partial\Psi}{\partial x}dx \qquad v = -\Psi^*$

$$\therefore \frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int\Psi^*\frac{\partial\Psi}{\partial x}dx$$

<u>EXPECTATION VALUE OF MOMENTUM:</u>

$$\langle p\rangle = m\langle v\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\Big(\Psi^*\frac{\partial\Psi}{\partial x}\Big)dx$$

EQUIVALENTLY, $\langle x\rangle = \int\Psi^*(x)\,\Psi\,dx$

$$\langle p\rangle = \int\Psi^*\Big(\frac{\hbar}{i}\frac{\partial}{\partial x}\Big)\Psi\,dx$$

NOMENCLATURE: - POSITION OPERATOR IS x

- MOMENTUM OPERATOR IS $\frac{\hbar}{i}\frac{\partial}{\partial x}$

$\therefore$ THE EXPECTATION VALUE OF ANY QUANTITY $Q(x,p)$ IS:

$$\langle Q(x,p)\rangle = \int\Psi^*\,Q\Big(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\Big)\,\Psi\,dx$$

FOR EXAMPLE, THE EXPECTATION VALUE OF KINETIC ENERGY IS:

$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{-\hbar^2}{2m}\int\Psi^*\frac{\partial^2\Psi}{\partial x^2}dx$$

$\boxed{\text{CONSIDER FIGS. 1.7 AND 1.8.}}$

RIGOROUS FOURIER ANALYSIS CONFIRMS THE QUALITATIVE RESULT IN FIGS. 1.7 AND 1.8 THAT THE WAVELENGTH AND POSITION OF A WAVE CANNOT BE PERFECTLY DEFINED CONCURRENTLY.

OF COURSE, THIS UNCERTAINTY ALSO APPLIES TO Ψ.

FURTHERMORE, $p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$ (DE BROGLIE FORMULA)

$\therefore$ SPREAD IN WAVELENGTH $\Rightarrow$ SPREAD IN MOMENTUM

QUANTITATIVELY, $\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$ $\leftarrow$ HEISENBERG UNCERTAINTY PRINCIPLE

$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ (STANDARD DEVIATION IN x)

$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ (STANDARD DEVIATION IN p)

<u>NOTE</u>: UNCERTAINTY PRINCIPLE HOLDS FOR OTHER INCOMPATIBLE OBSERVABLES (e.g., ENERGY AND TIME)

<u>NOTE</u>: EVERYTHING IN THIS LECTURE CAN BE PLACED ON FIRMER THEORETICAL FOOTING USING THE MATRIX FORMULATION OF QUANTUM MECHANICS. IF YOU ARE INTERESTED, PLEASE SEE CHAPTER 3 OF GRIFFITHS.

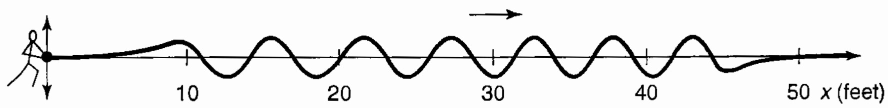

FIGURE 1.7: A wave with a (fairly) well-defined *wavelength*, but an ill-defined *position*.

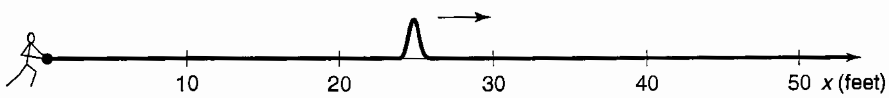

FIGURE 1.8: A wave with a (fairly) well-defined *position*, but an ill-defined *wavelength*.