LECTURE #3

CONSIDER A TIME-INDEPENDENT POTENTIAL: $V(x)$

SCH. EQ.: $i\hbar \dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \Psi}{\partial x^2} + V\Psi$

ATTEMPT TO SOLVE USING SEPARATION OF VARIABLES:

ASSUME: $\Psi(x,t) = \psi(x)\,\phi(t)$

$$\therefore \quad i\hbar \psi \frac{d\phi}{dt} = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2}\phi + V\psi\phi$$

DIVIDE BOTH SIDES BY $\psi\phi$:

$$\underbrace{i\hbar \frac{1}{\phi} \frac{d\phi}{dt}}_{\text{ONLY A FUNCTION OF } t} = \underbrace{-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V}_{\text{ONLY A FUNCTION OF } x}$$

SINCE t AND x ARE INDEPENDENT VARIABLES, BOTH SIDES OF THIS EQUATION MUST BE CONSTANT.

i.e., $\quad i\hbar \dfrac{1}{\phi}\dfrac{d\phi}{dt} = E \Rightarrow \dfrac{d\phi}{dt} = -\dfrac{iE}{\hbar}\phi \Rightarrow \phi = e^{-iEt/\hbar}$

NOTE: THE INTEGRATION CONSTANT IS NOT INCLUDED SINCE IT CAN JUST BE ABSORBED INTO ψ.

ALSO, $\quad -\dfrac{\hbar^2}{2m}\dfrac{1}{\psi}\dfrac{d^2\psi}{dx^2} + V = E \Rightarrow -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V\psi = E\psi$

$\therefore$ FOR ANY TIME-INDEPENDENT POTENTIAL $V(x)$,

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

WHERE $\psi(x)$ IS THE SOLUTION TO THE TIME-INDEPENDENT SCHRÖDINGER EQUATION:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi}$$

PROPERTIES OF THESE SEPARABLE SOLUTIONS:

(1) STATIONARY STATES:

ALTHOUGH $\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$ IS TIME-DEPENDENT,

$$\left[\, |\Psi(x,t)|^2 = \Psi^*\Psi = \psi^* e^{iEt/\hbar}\, \psi e^{-iEt/\hbar} = |\psi(x)|^2 \,\right]$$

THE PROBABILITY DENSITY IS TIME-INDEPENDENT.

FURTHERMORE, ALL EXPECTATION VALUES ARE TIME-INDEPENDENT:

$$\langle Q(x,p)\rangle = \int \psi^* \, Q\!\left(x, \frac{\hbar}{i}\frac{d}{dx}\right) \psi \, dx$$

$\Rightarrow \langle p \rangle = m \dfrac{d\langle x\rangle}{dt} = 0$ (THESE STATES ARE "STATIONARY")

(2) DEFINITE TOTAL ENERGY:

HAMILTONIAN OPERATOR: $\hat{H} = \dfrac{\hat{p}^2}{2m} + V = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V$

$\Rightarrow$ TIME-INDEPENDENT SCH. EQ. : $\hat{H}\psi = E\psi$

EXPECTATION VALUE OF TOTAL ENERGY:

$$\langle H \rangle = \int \psi^* \hat{H} \psi \, dx = E \int |\psi|^2 dx = E$$

$$\hat{H}^2 \psi = \hat{H}(\hat{H}\psi) = \hat{H}(E\psi) = E(\hat{H}\psi) = E^2 \psi$$

$$\therefore \langle H^2 \rangle = \int \psi^* \hat{H}^2 \psi \, dx = E^2 \int |\psi|^2 dx = E^2$$

$$\therefore \sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = E^2 - E^2 = 0$$

$\Rightarrow$ THE TOTAL ENERGY IS DEFINITE AND EQUAL TO E.

(3) THE GENERAL SOLUTION TO THE TIME-DEPENDENT SCH. EQ. IS A LINEAR COMBINATION OF SEPERABLE SOLUTIONS.

i.e., $$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}}$$ c_n: COMPLEX CONSTANTS

NOTE: $|\Psi(x,t)|^2$ IS TIME-DEPENDENT AND DOES NOT HAVE DEFINITE ENERGY

NOTE THE FOLLOWING GENERIC FEATURES OF $\psi_n(x)$:

(1) ORTHOGONAL: $\int dx \, \psi_m^*(x) \psi_n(x) = 0$ FOR $m \neq n$

OF COURSE, $\int dx \, \psi_m^*(x) \psi_n(x) = 1$ FOR $m = n$

(2) <u>COMPLETE</u>: ANY FUNCTION $f(x)$ WHICH SATISFIES THE BOUNDARY CONDITIONS CAN BE WRITTEN AS:

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$ (AS IS FAMILIAR FROM FOURIER ANALYSIS)

GIVEN $f(x)$, WE CAN CALCULATE c_n:

$$\int dx\, \psi_m^*(x) f(x) = \sum_{n=1}^{\infty} c_n \underbrace{\int dx\, \psi_m^*(x) \psi_n(x)}$$

KRONECKER DELTA $\longrightarrow$ $\delta_{mn} = \begin{cases} 0 & \text{FOR } m \neq n \\ 1 & \text{FOR } m = n \end{cases}$

$$\Rightarrow c_m = \int dx\, \psi_m^*(x) f(x)$$

IN PRACTICE, YOU ARE TYPICALLY GIVEN $V(x)$ AND $\Psi(x,0)$

<u>STEP #1</u>: SOLVE TIME-INDEPENDENT SCH. EQ. TO FIND:

$\psi_1(x), \psi_2(x), \psi_3(x), \ldots$

AND $E_1, E_2, E_3, \ldots$

<u>STEP #2</u>: GENERAL SOLUTION: $\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$

$$\Rightarrow \Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x) \Rightarrow c_m = \int dx\, \psi_m^*(x) \Psi(x,0)$$

ONCE YOU HAVE ALL OF THE c_m, YOU'RE DONE! SIMPLY PLUG THEM BACK INTO THE GENERAL SOLUTION.

$|c_n|^2$ IS THE PROBABILITY THAT A MEASUREMENT OF ENERGY WOULD YIELD THE VALUE E_n

$$\therefore \boxed{\sum_{n=1}^{\infty} |c_n|^2 = 1}$$

(THIS FOLLOWS DIRECTLY FROM THE NORMALIZATION OF Ψ)

FURTHERMORE,

$$\boxed{\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n}$$

(THIS FOLLOWS DIRECTLY FROM THE TIME-INDEPENDENT SCH. EQ.)

<u>NOTE</u>: $\langle H \rangle$ IS INDEPENDENT OF TIME. THIS IS A MANIFESTATION OF CONSERVATION OF ENERGY IN QUANTUM MECHANICS.