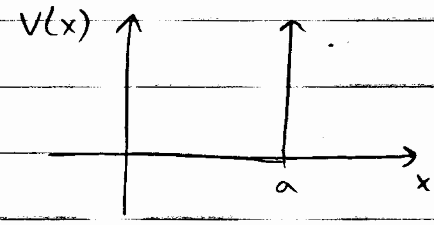
LECTURE #5

CONSIDER THE INFINITE SQUARE WELL:



i.e., $V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{OTHERWISE} \end{cases}$

EVIDENTLY, THE PARTICLE IS FORBIDDEN TO LEAVE THE WELL

$\Rightarrow \psi(x) = 0$ FOR $x \le 0$ OR $x \ge a$

INSIDE THE WELL (i.e., $0 \le x \le a$):

TIME-INDEPENDENT SCH. EQ.: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E\psi$

i.e., $\dfrac{d^2\psi}{dx^2} = -k^2\psi$ WHERE $k = \dfrac{\sqrt{2mE}}{\hbar}$, $E > 0$

$\Rightarrow \psi(x) = A\sin kx + B\cos kx$

WHERE A AND B ARE CONSTANTS DETERMINED FROM THE BOUNDARY CONDITIONS

CONTINUITY OF $\psi(x) \Rightarrow \psi(0) = \psi(a) = 0$

$\psi(0) = B = 0 \Rightarrow \psi(x) = A\sin kx$

$\psi(a) = A\sin ka = 0 \Rightarrow ka = \pi, 2\pi, 3\pi, \ldots$

i.e., $k_n = \dfrac{n\pi}{a}$, $n = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad n = 1, 2, \ldots}$$

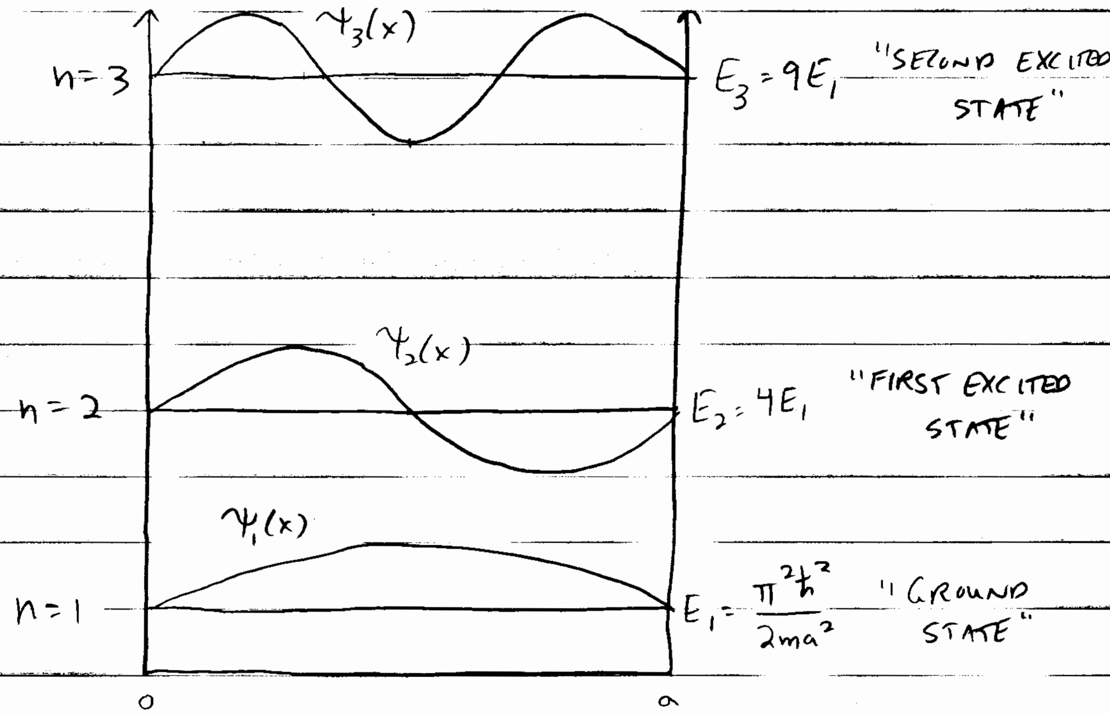
* THE BOUNDARY CONDITIONS IMPLY QUANTIZATION OF ENERGY

TO FIND A, WE NORMALIZE ψ:

$$1 = \int_0^a |A|^2 \sin^2(kx)\,dx = |A|^2 \frac{a}{2} \Rightarrow A = \sqrt{\frac{2}{a}}$$

$$\therefore \boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \quad n = 1, 2, \ldots}$$

PICTORIALLY,



NOTE: (1) $\psi_1(x), \psi_3(x), \ldots$ ARE EVEN WITH RESPECT TO THE CENTER OF THE WELL; $\psi_2(x), \psi_4(x), \ldots$ ARE ODD.

(2) MORE NODES $\Rightarrow$ MORE CURVATURE $\Rightarrow$ MORE ENERGY (CONSISTENT WITH SCH. EQ.)

(3) ENERGY INCREASES WITH DECREASING WIDTH (CONSISTENT WITH UNCERTAINTY PRINCIPLE)

CONSIDER THE HARMONIC OSCILLATOR:

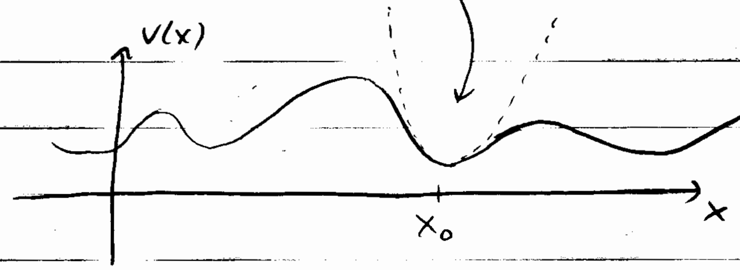
CLASSICALLY, THIS DESCRIBES A MASS m ON A SPRING WITH SPRING CONSTANT K

MOTION IS GOVERNED BY HOOKE'S LAW: $F = -kx = m \frac{d^2x}{dt^2}$

$\Rightarrow x(t) = A\sin(\omega t) + B\cos(\omega t)$ WHERE $\omega = \sqrt{\frac{k}{m}}$

THE POTENTIAL ENERGY IS: $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$

WHILE THERE'S NO SUCH THING AS A PERFECT HARMONIC OSCILLATOR, NEARLY ANY POTENTIAL NEAR A MINIMUM IS APPROXIMATELY PARABOLIC



TAYLOR SERIES EXPAND ABOUT THE MINIMUM:

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \cdots$$

SINCE x_0 IS A MINIMUM, $V'(x_0) = 0$

FURTHERMORE, $V(x_0)$ IS SIMPLY A CONSTANT OFFSET.

$$\therefore \; V(x) \simeq \frac{1}{2}V''(x_0)(x-x_0)^2$$

$\Rightarrow$ VIRTUALLY ANY OSCILLATORY MOTION IS APPROXIMATELY HARMONIC FOR SMALL OSCILLATION AMPLITUDES!

QUANTUM MECHANICALLY, THE TIME-IND. SCH. EQ. IS:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

TWO METHODS TO SOLVE THIS DIFFERENTIAL EQUATION:

(1) POWER SERIES METHOD: "BRUTE FORCE" SOLUTION WHICH WE WILL SKIP FOR NOW SINCE WE WILL LEARN THIS APPROACH LATER FOR THE COULOMB POTENTIAL

(2) LADDER OPERATORS: ALGEBRAIC METHOD THAT IS QUICKER AND SIMPLER:

REWRITE SCH. EQ. IN TERMS OF THE MOMENTUM OPERATOR:

$$\underbrace{\frac{1}{2m}\left[p^2+(m\omega x)^2\right]}_{H}\psi = E\psi$$

THE PLAN IS TO FACTOR THE HAMILTONIAN, WHICH WOULD BE EASY IF IT WERE MADE UP OF NUMBERS SINCE

$$u^2+v^2=(iu+v)(-iu+v)$$

HOWEVER, OPERATORS DO NOT, IN GENERAL, COMMUTE:

i.e., $xp \neq px$

NEVERTHELESS, WE WILL EXPLORE THE FOLLOWING OPERATORS:

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp ip + m\omega x\right)$$

$$a_- a_+ = \frac{1}{2\hbar m\omega}(ip+m\omega x)(-ip+m\omega x)$$
$$= \frac{1}{2\hbar m\omega}\left(p^2+(m\omega x)^2 - im\omega(xp-px)\right)$$

DEFINITION: THE COMMUTATOR OF OPERATORS A AND B IS:

$$[A,B] = AB - BA$$

$$\therefore \; a_- a_+ = \frac{1}{2\hbar m\omega}\left(p^2 + (m\omega x)^2\right) - \frac{i}{2\hbar}[x,p]$$

WE SEEK $[x,p]$ BY HAVING IT ACT ON A TEST FUNCTION:

$$[x,p]f(x) = \left[x\,\frac{\hbar}{i}\frac{d}{dx}f - \frac{\hbar}{i}\frac{d}{dx}(xf)\right]$$

$$= \frac{\hbar}{i}\left(x\frac{df}{dx} - x\frac{df}{dx} - f\right) = i\hbar f(x)$$

$$\therefore \; \boxed{[x,p] = i\hbar} \quad \text{CANONICAL COMMUTATION RELATION}$$

$$\therefore \; a_- a_+ = \frac{1}{\hbar\omega}H + \frac{1}{2}$$

$$\Rightarrow \; H = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)$$

REPEATING THIS ANALYSIS FOR $a_+ a_-$ YIELDS:

$$a_+ a_- = \frac{1}{\hbar\omega}H - \frac{1}{2} \;\Rightarrow\; H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

NOTE: $[a_-, a_+] = a_- a_+ - a_+ a_- = 1$

$\therefore$ THE SCH. EQ. CAN BE WRITTEN AS:

$$\hbar\omega\left(a_\pm a_\mp \pm \frac{1}{2}\right)\psi = E\psi$$

CONSIDER $H(a_+\psi)$:

$$H(a_+\psi) = \hbar\omega\left(a_+a_- + \tfrac{1}{2}\right)(a_+\psi) = \hbar\omega\left(a_+a_-a_+ + \tfrac{1}{2}a_+\right)\psi$$

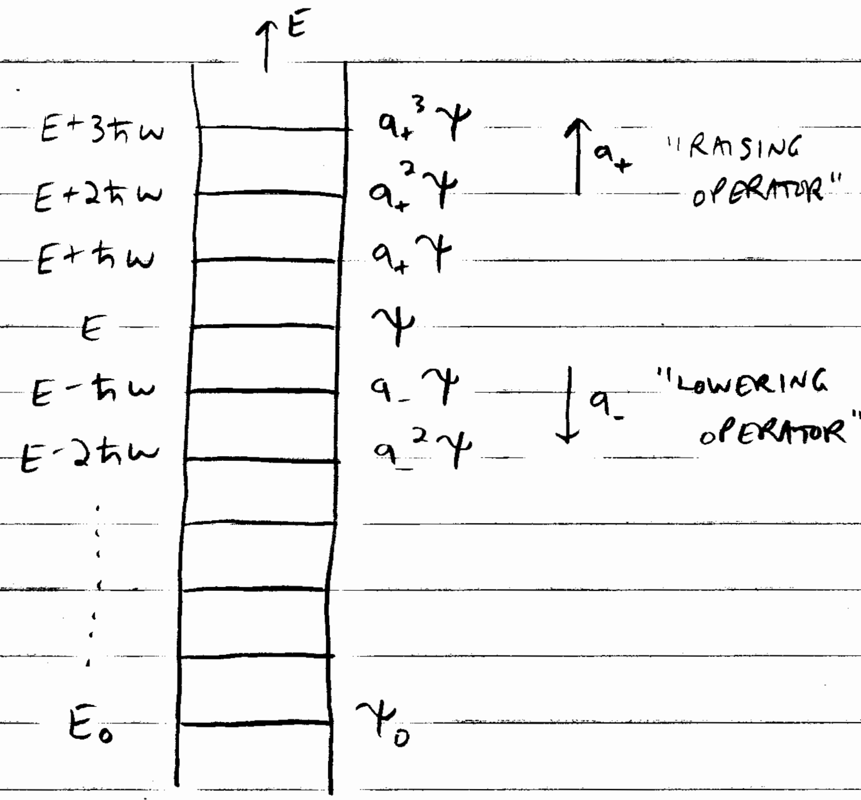
$$= \hbar\omega\, a_+\left(a_-a_+ + \tfrac{1}{2}\right)\psi = a_+\left[\hbar\omega\left(a_+a_- + 1 + \tfrac{1}{2}\right)\psi\right]$$

$$= a_+(H + \hbar\omega)\psi = a_+(E+\hbar\omega)\psi = (E+\hbar\omega)(a_+\psi)$$

i.e., $(a_+\psi)$ SATISFIES THE SCH. EQ. WITH ENERGY $E+\hbar\omega$

REPEATING THIS ANALYSIS FOR $(a_-\psi)$ SHOWS THAT $(a_-\psi)$ ALSO SATISFIES THE SCH. EQ. WITH ENERGY $E-\hbar\omega$

"LADDER" OF STATES:



FOR THE LOWEST RUNG, $a_-\psi_0 = 0$

$$\therefore \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0$$

$$\therefore \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x \psi_0$$

$$\int \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \Rightarrow \ln \psi_0 = -\frac{m\omega}{2\hbar}x^2 + \text{CONSTANT}$$

$$\therefore \psi_0 = A e^{-\frac{m\omega}{2\hbar}x^2}$$

NORMALIZE: $1 = |A|^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$

$$\therefore A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \Rightarrow \boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

E_0 IS DETERMINED FROM THE SCH. EQ.: $H\psi_0 = E_0\psi_0$

$$\hbar\omega\left(a_+a_- + \frac{1}{2}\right)\psi_0 = \frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0 \Rightarrow E_0 = \frac{\hbar\omega}{2}$$

$$\therefore \boxed{\psi_n(x) = A_n (a_+)^n \psi_0(x) \quad \text{WITH} \quad E_n = \left(n+\frac{1}{2}\right)\hbar\omega}$$

NOTE: $n = 0, 1, 2, \ldots$

NOTE: WITH ADDITIONAL ALGEBRA, IT CAN BE SHOWN THAT $A_n = \frac{1}{\sqrt{n!}}$

(SEE GRIFFITHS SECTION 2.3 FOR DETAILS)

FIG. 2.7 → GRAPHICAL DEPICTIONS OF STATIONARY STATE SOLUTIONS OF THE HARMONIC OSCILLATOR

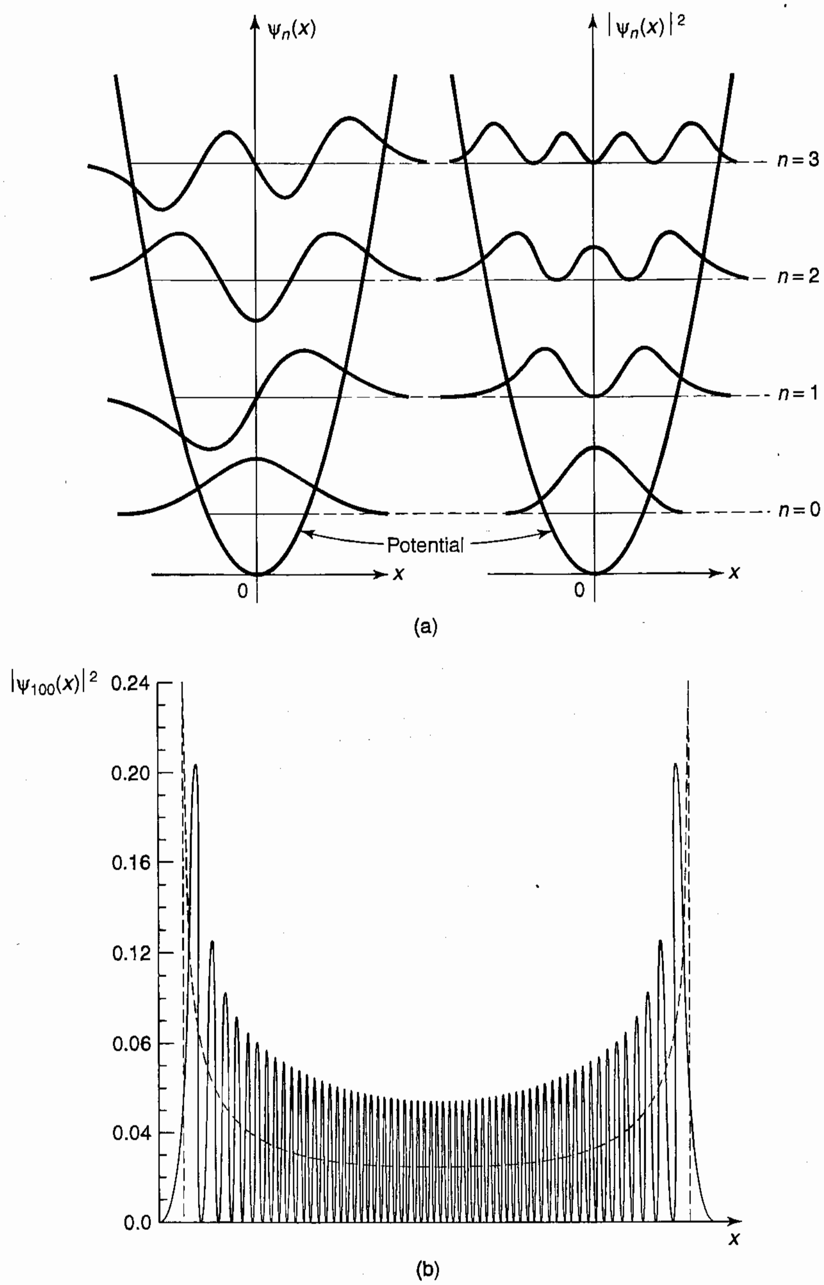


FIGURE 2.7: (a) The first four stationary states of the harmonic oscillator. This material is used by permission of John Wiley & Sons, Inc.; Stephen Gasiorowicz, *Quantum Physics*, John Wiley & Sons, Inc., 1974. (b) Graph of $|\psi_{100}|^2$, with the classical distribution (dashed curve) superimposed.