LECTURE #15

CONSIDER AN ARBITRARY POTENTIAL FOR WHICH ONE PARTICLE ENERGIES ARE $E_1, E_2, E_3, \ldots$ WITH DEGENERACIES $d_1, d_2, d_3, \ldots$

IF WE PUT N PARTICLES INTO THIS POTENTIAL, HOW MANY DIFFERENT WAYS CAN A PARTICULAR CONFIGURATION $(N_1, N_2, N_3, \ldots)$ BE ACHIEVED?

THE ANSWER, $Q(N_1, N_2, N_3, \ldots)$ DEPENDS ON WHETHER THE PARTICLES ARE DISTINGUISHABLE, IDENTICAL FERMIONS, OR IDENTICAL BOSONS.

DISTINGUISHABLE PARTICLES:

HOW MANY WAYS CAN WE SELECT N_1 PARTICLES (FROM N TOTAL) TO BE PLACED IN THE FIRST ENERGY?

ANSWER: THE BINOMIAL COEFFICIENT, "N CHOOSE N_1"

$$\binom{N}{N_1} = \frac{N!}{N_1!\,(N-N_1)!}$$

HOWEVER, EACH OF THESE PARTICLES HAS d_1 CHOICES:

$$\therefore \text{TOTAL \# OF POSSIBILITIES} = \frac{N!\,d_1^{N_1}}{N_1!\,(N-N_1)!}$$

FOR THE SECOND ENERGY, THE ANALYSIS IS THE SAME EXCEPT THAT THERE ARE ONLY $(N-N_1)$ PARTICLES LEFT TO WORK WITH:

$$\frac{(N-N_1)!\, d_2^{N_2}}{N_2!\,(N-N_1-N_2)!} \cdots$$

$\therefore\ Q(N_1, N_2, N_3, \ldots) =$

$$\frac{N!\, d_1^{N_1}}{N_1!(N-N_1)!} \frac{(N-N_1)!\, d_2^{N_2}}{N_2!(N-N_1-N_2)!} \frac{(N-N_1-N_2)!\, d_3^{N_3}}{N_3!(N-N_1-N_2-N_3)!} \cdots$$

$$= N! \frac{d_1^{N_1} d_2^{N_2} d_3^{N_3}}{N_1!\, N_2!\, N_3!} \cdots = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$$

IDENTICAL FERMIONS

(1) ANTISYMMETRIZATION $\Rightarrow$ THERE IS ONLY ONE N-PARTICLE STATE IN WHICH A SET OF ONE PARTICLE STATES IS OCCUPIED

(2) PAULI EXCLUSION $\Rightarrow$ ONLY ONE PARTICLE CAN OCCUPY ANY GIVEN STATE

$\therefore$ THERE ARE $\binom{d_n}{N_n}$ WAYS TO CHOOSE N_n OCCUPIED STATES FOR ENERGY E_n

$$\therefore\ Q(N_1, N_2, N_3, \ldots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!\,(d_n - N_n)!}$$

IDENTICAL BOSONS:

SAME AS FERMIONS EXCEPT THAT THERE IS NO RESTRICTION ON THE NUMBER OF PARTICLES THAT CAN SHARE THE SAME ONE PARTICLE STATE.

i.e., HOW MANY WAYS CAN WE PUT N_n IDENTICAL PARTICLES INTO d_n BASKETS?

<u>ANSWER</u>: $\binom{N_n + d_n - 1}{N_n}$

$$\therefore\ Q(N, N_2, N_3, \ldots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$$

<u>RECALL</u>: THE MOST PROBABLE CONFIGURATION IS THE ONLY STATISTICALLY RELEVANT CONFIGURATION.

$\therefore$ WE WANT TO MAXIMIZE $Q(N_1, N_2, N_3, \ldots)$ SUBJECT TO THE FOLLOWING CONSTRAINTS:

(1) $\sum_{n=1}^{\infty} N_n = N$ (CONSERVATION OF PARTICLES)

(2) $\sum_{n=1}^{\infty} N_n E_n = E$ (CONSERVATION OF ENERGY)

THIS PROBLEM IS MOST CONVENIENTLY HANDLED BY THE METHOD OF <u>LAGRANGE MULTIPLIERS</u>.

LET $G = \ln Q + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$

LAGRANGE MULTIPLIERS

Q IS MAXIMIZED WHEN $\frac{\partial G}{\partial N_n} = 0$, $\frac{\partial G}{\partial \alpha} = 0$, $\frac{\partial G}{\partial \beta} = 0$

<u>NOTE</u>: $\frac{\partial G}{\partial \alpha} = 0$ AND $\frac{\partial G}{\partial \beta} = 0$ ARE ALREADY SATISFIED BY THE PHYSICAL CONSTRAINTS

DISTINGUISHABLE PARTICLES:

$$G = \ln(N!) + \sum_{n=1}^{\infty}\left[N_n \ln(d_n) - \ln(N_n!)\right] + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

INVOKE STERLING'S APPROXIMATION: $\ln(z!) \approx z\ln(z) - z$ FOR $z \gg 1$ OR $z=0$

$$\therefore\ G \simeq \sum_{n=1}^{\infty}\left[N_n \ln(d_n) - N_n \ln(N_n) + N_n - \alpha N_n - \beta E_n N_n\right] + \ln(N!) + \alpha N + \beta E$$

$$\therefore\ \frac{\partial G}{\partial N_n} = \ln(d_n) - N_n \frac{1}{N_n} - \ln(N_n) + 1 - \alpha - \beta E_n$$

$$= \ln(d_n) - \ln(N_n) - \alpha - \beta E_n = 0$$

$$\Rightarrow\ N_n = d_n e^{-(\alpha + \beta E_n)}$$

MOST PROBABLE OCCUPATION NUMBERS FOR DISTINGUISHABLE PARTICLES

IDENTICAL FERMIONS:

$$G = \sum_{n=1}^{\infty}\left(\ln(d_n!) - \ln(N_n!) - \ln\left[(d_n - N_n)!\right]\right) + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

ASSUME THAT $N_n \gg 1$ AND $d_n \gg N_n$ (FINITE TEMP.) OR $d_n = N_n$ ($T = 0K$),

$$\therefore G \simeq \sum_{n=1}^{\infty} [\ln(d_n!) - N_n \ln(N_n) + N_n - (d_n - N_n)\ln(d_n - N_n) + (d_n - N_n) - \alpha N_n - \beta E_n N_n] + \alpha N + \beta E$$

$$\therefore \frac{\partial G}{\partial N_n} = -\ln(N_n) + \ln(d_n - N_n) - \alpha - \beta E_n = 0$$

$$\Rightarrow N_n = \frac{d_n}{e^{(\alpha + \beta E_n)} + 1}$$

MOST PROBABLE OCCUPATION NUMBERS FOR IDENTICAL FERMIONS

IDENTICAL BOSONS:

$$G = \sum_{n=1}^{\infty} [\ln[(N_n + d_n - 1)!] - \ln(N_n!) - \ln[(d_n - 1)!]] + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

$$\therefore G \simeq \sum_{n=1}^{\infty} \left[(N_n + d_n - 1)\ln(N_n + d_n - 1) - (N_n + d_n - 1) - N_n \ln(N_n) + N_n - \ln[(d_n - 1)!] - \alpha N_n - \beta E_n N_n\right] + \alpha N + \beta E$$

$$\therefore \frac{\partial G}{\partial N_n} = \ln(N_n + d_n - 1) - \ln(N_n) - \alpha - \beta E_n = 0$$

$$\Rightarrow N_n = \frac{d_n - 1}{e^{(\alpha + \beta E_n)} - 1} \simeq \frac{d_n}{e^{(\alpha + \beta E_n)} - 1}$$

MOST PROBABLE OCCUPATION NUMBERS FOR IDENTICAL BOSONS

DEFINE: (1) TEMPERATURE: $T = \frac{1}{K_B \beta} \Rightarrow \beta = \frac{1}{K_B T}$

K_B = BOLTZMANN'S CONSTANT = 1.381×10^{-23} J/K

(2) CHEMICAL POTENTIAL: $\mu(T) = -\alpha K_B T$

∴ THE MOST PROBABLE NUMBER OF PARTICLES IN A PARTICULAR (ONE-PARTICLE) STATE WITH ENERGY E IS:

$$f(E) = e^{-(E-\mu)/k_B T}$$

MAXWELL-BOLTZMANN DISTRIBUTION FOR DISTINGUISHABLE PARTICLES (e.g., IDEAL GAS)

$$f(E) = \frac{1}{e^{(E-\mu)/k_B T}+1}$$

FERMI-DIRAC DISTRIBUTION FOR IDENTICAL FERMIONS (e.g., ELECTRONS IN A SOLID)

$$f(E) = \frac{1}{e^{(E-\mu)/k_B T}-1}$$

BOSE EINSTEIN DISTRIBUTION FOR IDENTICAL BOSONS (e.g., PHOTONS, PHONONS)

CONSIDER FERMI-DIRAC DISTRIBUTION: