LECTURE #21

THIS LECTURE IS BASED ON:

D. J. GRIFFITHS AND C. A. STEINKE, "WAVES IN LOCALLY PERIODIC MEDIA," AM. J. PHYS., 69, 137–154 (2001).

THUS FAR, WE HAVE STUDIED THREE TYPES OF WAVE MOTION:

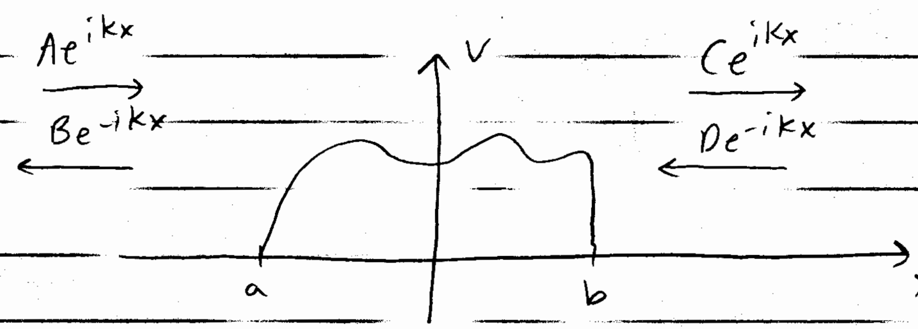
(1) TRAVELING WAVES: CAN HAVE ANY FREQUENCY (e.g., SCATTERING STATES)

(2) STANDING WAVES: OCCUR ONLY FOR DISCRETE "ALLOWED" FREQUENCIES (e.g., BOUND STATES)

(3) INFINITE PERIODIC MEDIA: FREQUENCIES FALL INTO CONTINUOUS "BANDS" SEPARATED BY FORBIDDEN "GAPS" (e.g., KRONIG-PENNEY)

WHAT ABOUT FINITE, LOCALLY PERIODIC MEDIA?

FIRST CONSIDER SCATTERING FROM AN ARBITRARY POTENTIAL:



THE GENERAL SOLUTION IS:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x<a \\ \psi_{ab}(x), & a<x<b \\ Ce^{ikx} + De^{-ikx}, & x>b \end{cases}$$

WHERE $k = \dfrac{\sqrt{2mE}}{\hbar}$

TO COMPLETE THE PROBLEM, YOU SOLVE THE SCH. EQ. FOR $a<x<b$ AND THEN INVOKE APPROPRIATE BOUNDARY CONDITIONS AT $x=a$ AND $x=b$.

THIS APPROACH YIELDS TWO LINEAR RELATIONS AMONG THE COEFFICIENTS A, B, C, AND D.

FOR THIS ANALYSIS, IT IS CONVENIENT TO EXPRESS A AND B IN TERMS OF C AND D:

$$\begin{pmatrix} A \\ B \end{pmatrix} = [M] \begin{pmatrix} C \\ D \end{pmatrix}$$

WHERE $[M] = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ IS CALLED THE "TRANSFER MATRIX"

WE CAN IMPOSE STRONG CONDITIONS ON $[M]$ BY RECOGNIZING:

(1) <u>TIME REVERSAL INVARIANCE</u>:

TIME-DEPENDENT SCH. EQ.:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

WHERE WE ASSUME THAT $V(x)$ IS TIME-INDEPENDENT AND REAL.

TAKE THE COMPLEX CONJUGATE AND REPLACE t WITH $-t$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi^*(x,-t) + V(x)\Psi^*(x,-t) = i\hbar\frac{\partial}{\partial t}\Psi^*(x,-t)$$

THIS IS THE SAME FORM AS THE TIME-DEPENDENT SCH. EQ.

$\therefore$ IF $\Psi(x,t)$ IS A SOLUTION, THEN SO IS $\Psi^*(x,-t)$

$$\therefore \psi^*(x) = \begin{cases} A^* e^{-ikx} + B^* e^{ikx}, & x<a \\ \psi^*_{ab}(x), & a<x<b \\ C^* e^{-ikx} + D^* e^{ikx}, & x>b \end{cases}$$

$\Rightarrow$ THIS EXCHANGES INCOMING AND OUTGOING WAVES.

$$\therefore \begin{pmatrix} B^* \\ A^* \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} D^* \\ C^* \end{pmatrix}$$

$$\therefore \begin{aligned} B^* &= M_{11} D^* + M_{12} C^* \Rightarrow B = M_{11}^* D + M_{12}^* C \\ A^* &= M_{21} D^* + M_{22} C^* \Rightarrow A = M_{21}^* D + M_{22}^* C \end{aligned}$$

$$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M_{22}^* & M_{21}^* \\ M_{12}^* & M_{11}^* \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

$$\therefore \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M_{22}^* & M_{21}^* \\ M_{12}^* & M_{11}^* \end{pmatrix}$$

$$\Rightarrow \boxed{M_{11} = M_{22}^*} \quad \text{AND} \quad \boxed{M_{21} = M_{12}^*}$$

(2) Conservation of Probability

In quantum mechanics in 1-D, the probability current:

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right),$$ is independent of x.

$$\therefore\ j|_{x<a} = j|_{x>b}$$

$$j|_{x<a} = \frac{\hbar}{2mi}\Big[\left(A^* e^{-ikx} + B^* e^{ikx}\right)(ik)\left(Ae^{ikx} - Be^{-ikx}\right)$$

$$-(ik)\left(B^* e^{ikx} - A^* e^{-ikx}\right)\left(Ae^{ikx} + Be^{-ikx}\right)\Big]$$

$$= \frac{\hbar k}{2m}\Big[|A|^2 + AB^* e^{2ikx} - BA^* e^{-2ikx} - |B|^2 - AB^* e^{2ikx}$$
$$+ |A|^2 - |B|^2 + BA^* e^{-2ikx}\Big]$$

$$= \frac{\hbar k}{m}\left[|A|^2 - |B|^2\right]$$

Similarly, $j|_{x>b} = \frac{\hbar k}{m}\left[|C|^2 - |D|^2\right]$

$$\therefore\ j|_{x<a} = j|_{x>b} \Rightarrow |A|^2 - |B|^2 = |C|^2 - |D|^2$$

In matrix notation,

$$\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A^* \\ B^* \end{pmatrix} = \begin{pmatrix} C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} C^* \\ D^* \end{pmatrix}$$

But $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$ (definition of the transfer matrix)

$$\therefore \begin{pmatrix} C & D \end{pmatrix}\begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{pmatrix}\begin{pmatrix} C^* \\ D^* \end{pmatrix}$$

$$= \begin{pmatrix} C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} C^* \\ D^* \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}\begin{pmatrix} M_{11}^* & M_{12}^* \\ -M_{21}^* & -M_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} |M_{11}|^2 - |M_{21}|^2 & M_{11}M_{12}^* - M_{21}M_{22}^* \\ M_{11}^*M_{12} - M_{22}M_{21}^* & |M_{12}|^2 - |M_{22}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

IN LIGHT OF OUR PREVIOUS CONSTRAINTS,
(i.e., $M_{11} = M_{22}^*$ AND $M_{21} = M_{12}^*$)
THIS MATRIX EQUATION YIELDS ONE ADDITIONAL CONSTRAINT:

$$|M_{11}|^2 - |M_{21}|^2 = 1$$

$\therefore$ IF WE LET $M_{11} = W$ AND $M_{12} = Z$,

$$\boxed{M = \begin{pmatrix} W & Z \\ Z^* & W^* \end{pmatrix} \text{ SUCH THAT } |W|^2 - |Z|^2 = 1}$$