LECTURE #6

TWO TYPES OF STATES IN QUANTUM MECHANICS:

(1) BOUND STATES: (e.g., INFINITE SQUARE WELL, HARMONIC OSCILLATOR)
   (a) PARTICLE IS LOCALIZED
   (b) Ψ IS NORMALIZABLE
   (c) Ψ IS LABELED BY A DISCRETE INDEX n

(2) SCATTERING STATES: (e.g., FREE PARTICLE WAVE PACKET)
   (a) PARTICLE IS PROPAGATING
   (b) Ψ IS NOT NORMALIZABLE
   (c) Ψ IS LABELED BY A CONTINUOUS VARIABLE k

* SOME POTENTIALS CAN SUPPORT BOTH TYPES OF STATES DEPENDING ON THE ENERGY OF THE PARTICLE.

$$\left.\begin{array}{l} E < [V(-\infty) \text{ AND } V(+\infty)] \Rightarrow \text{BOUND STATE} \\ E > [V(-\infty) \text{ AND } V(+\infty)] \Rightarrow \text{SCATTERING STATE} \end{array}\right\} \boxed{\text{FIG. 2.12}}$$

INTRODUCE THE DIRAC DELTA FUNCTION:

$$\delta(x) = \begin{cases} 0, & \text{IF } x \neq 0 \\ \infty, & \text{IF } x = 0 \end{cases} \quad \text{WITH} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

NOTE: $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

$$\therefore \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a)$$

$\Rightarrow$ THE DIRAC DELTA FUNCTION IS A "SAMPLING" FUNCTION

CONSIDER THE DELTA FUNCTION WELL: $V(x) = -\alpha\,\delta(x)$

WHERE $\alpha > 0$

SCH. EQ.:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi$$

THIS POTENTIAL YIELDS BOTH BOUND STATES ($E<0$) AND SCATTERING STATES ($E>0$).

FIRST CONSIDER BOUND STATES ($E<0$):

FOR $x<0$, $V(x)=0 \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2} = K^2\psi$

WHERE $K = \frac{\sqrt{-2mE}}{\hbar}$ (WHICH IS REAL AND POSITIVE SINCE $E<0$)

$$\Rightarrow \psi(x) = Ae^{-Kx} + Be^{Kx}$$

HOWEVER, THE FIRST TERM "BLOWS UP" AS $x \to -\infty \Rightarrow A=0$

$$\therefore \quad \psi(x) = Be^{Kx}, \quad x<0$$

AN ANALOGOUS ANALYSIS FOR $x>0 \Rightarrow \psi(x) = Fe^{-Kx}, \quad x>0$

WE NOW MUST STITCH THESE SOLUTIONS TOGETHER USING APPROPRIATE BOUNDARY CONDITIONS

ψ CONTINUOUS AT $x=0 \Rightarrow B=F \Rightarrow \psi(x) = \begin{cases} Be^{Kx}, & x\le 0 \\ Be^{-Kx}, & x\ge 0 \end{cases}$

SINCE V IS INFINITE AT $x=0$, THE BOUNDARY CONDITION FOR $\frac{d\psi}{dx}$ IS NOT OBVIOUS:

TRY INTEGRATING THE SCH. EQ. FROM $-\varepsilon$ TO $+\varepsilon$ AND THEN TAKE THE LIMIT AS $\varepsilon \to 0$:

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2\psi}{dx^2}dx + \int_{-\varepsilon}^{\varepsilon} V(x)\psi(x)\,dx = E\int_{-\varepsilon}^{+\varepsilon}\psi(x)dx$$

IN THE LIMIT AS $\varepsilon \to 0$, THE RIGHT HAND SIDE GOES TO ZERO SINCE IT REPRESENTS THE AREA OF A SLIVER WITH FINITE HEIGHT AND VANISHING WIDTH.

$$\therefore \Delta\left(\frac{d\psi}{dx}\right) = \lim_{\varepsilon\to 0}\left(\frac{d\psi}{dx}\Big|_{+\varepsilon} - \frac{d\psi}{dx}\Big|_{-\varepsilon}\right) = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\varepsilon} V(x)\psi(x)dx$$

NORMALLY, THE LIMIT ON THE RIGHT IS AGAIN ZERO, WHICH IS WHY $\frac{d\psi}{dx}$ IS NORMALLY CONTINUOUS. HOWEVER, IF $V(x)$ IS INFINITE, THIS ARGUMENT FAILS.

FOR THE CASE OF $V(x) = -\alpha\,\delta(x)$, $\quad \Delta\left(\frac{d\psi}{dx}\right) = \frac{-2m\alpha}{\hbar^2}\psi(0)$

$$\frac{d\psi}{dx} = \begin{cases} -BKe^{-Kx}, & x>0 \\ BKe^{Kx}, & x<0 \end{cases} \quad \begin{array}{l} \Rightarrow \frac{d\psi}{dx}\Big|_+ = -BK \\ \Rightarrow \frac{d\psi}{dx}\Big|_- = BK \end{array} \Bigg\} \Rightarrow \Delta\left(\frac{d\psi}{dx}\right) = -2BK$$

FURTHERMORE, $\psi(0) = B \;\Rightarrow\; -2BK = \frac{-2m\alpha}{\hbar^2}B \Rightarrow K = \frac{m\alpha}{\hbar^2}$

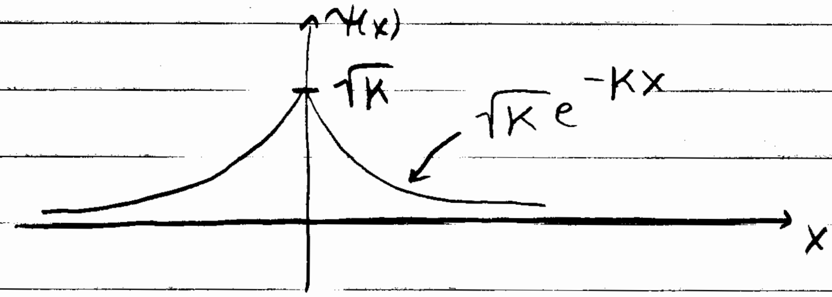
$$\therefore\; E = -\frac{\hbar^2K^2}{2m} = -\frac{m\alpha^2}{2\hbar^2} \;\Rightarrow\; \underline{\text{ONE}}\text{ BOUND STATE}$$

NORMALIZE ψ: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2|B|^2 \int_0^{\infty} e^{-2Kx} dx = \frac{|B|^2}{K} = 1$

$\therefore \quad B = \sqrt{K} = \frac{\sqrt{m\alpha}}{\hbar}$

$$\therefore \quad \boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \quad ; \quad E = -\frac{m\alpha^2}{2\hbar^2}}$$

↑ ONE BOUND STATE REGARDLESS OF ITS "STRENGTH" α



NOW CONSIDER SCATTERING STATES ($E > 0$).

$x<0$: $\quad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad , \quad k = \frac{\sqrt{2mE}}{\hbar}$

$\therefore \psi(x) = Ae^{ikx} + Be^{-ikx}$

$x<0$: $\quad \psi(x) = Fe^{ikx} + Ge^{-ikx}$

ψ CONTINUOUS AT $x=0$: $\quad \underline{A + B = F + G}$

$$\frac{d\psi}{dx} = \begin{cases} ik\left(Fe^{ikx} - Ge^{-ikx}\right), & x>0 \Rightarrow \left.\frac{d\psi}{dx}\right|_+ = ik(F-G) \\ ik\left(Ae^{ikx} - Be^{-ikx}\right), & x<0 \Rightarrow \left.\frac{d\psi}{dx}\right|_- = ik(A-B) \end{cases}$$

$\therefore \Delta\left(\frac{d\psi}{dx}\right) = ik\left(F - G - A + B\right) \quad ; \quad \psi(0) = A + B$

$\therefore$ 2nd BOUNDARY CONDITION (i.e. $\Delta\left(\frac{d\psi}{dx}\right) = \frac{-2m\alpha}{\hbar^2}\psi(0)$)

$$\Rightarrow \; ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$\therefore \quad \underline{F-G = A(1+2i\beta) - B(1-2i\beta)} \text{ WHERE } \beta = \frac{m\alpha}{\hbar^2 k}$$

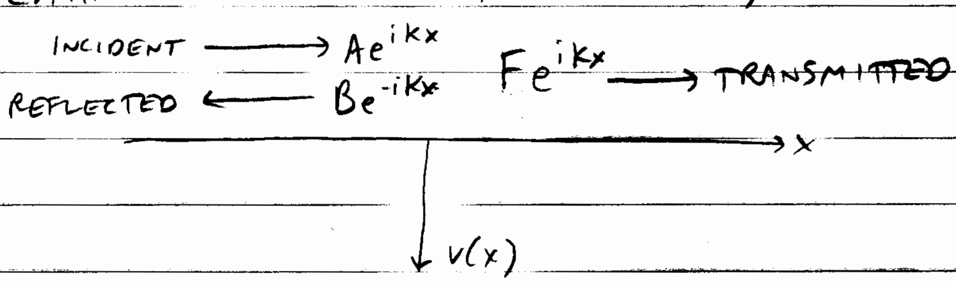
WE HAVE 2 EQUATIONS FOR 4 UNKNOWNS!

NORMALIZATION WON'T HELP SINCE SCATTERING STATES ARE NOT NORMALIZABLE.

PHYSICALLY, $x<0$: A IS AMPLITUDE OF WAVE MOVING RIGHT
B IS AMPLITUDE OF WAVE MOVING LEFT

$x>0$: F IS AMPLITUDE OF WAVE MOVING RIGHT
G IS AMPLITUDE OF WAVE MOVING LEFT

IN A TYPICAL EXPERIMENT, A WAVE FROM ONE DIRECTION IS INCIDENT ON THE SCATTERING POTENTIAL (ASSUME IT IS COMING FROM THE LEFT $\Rightarrow G=0$)



WITH $G=0$, THE PREVIOUS TWO EQUATIONS CAN BE REDUCED TO:

$$B = \frac{i\beta}{1-i\beta}A, \quad F = \frac{1}{1-i\beta}A$$

PROBABILITY IS GIVEN BY $|\Psi|^2$. THEREFORE, THE REFLECTION COEFFICIENT IS:

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} = \frac{1}{1+1/\beta^2}$$

SIMILARLY, THE TRANSMISSION COEFFICIENT IS:

$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}.$$

NOTE: $R+T=1$ (AS EXPECTED)

NOTE: $\beta^2 = \frac{m^2\alpha^2}{\hbar^4 K^2}$, $E = \frac{\hbar^2 K^2}{2m}$

$$\therefore \boxed{R = \frac{1}{1+(2\hbar^2 E/m\alpha^2)}, \quad T = \frac{1}{1+(m\alpha^2/2\hbar^2 E)}}$$

NOTE: AS $E\uparrow$, $T\uparrow$ AND $R\downarrow$ AS EXPECTED.

WHAT ABOUT THE DELTA-FUNCTION BARRIER? i.e., $V(x) = \alpha\,\delta(x)$

OBVIOUSLY, THERE IS NO BOUND STATE.

ON THE OTHER HAND, THE SCATTERING STATES ARE IDENTICAL SINCE R AND T ARE A FUNCTION OF α^2.

$\Rightarrow$ PARTICLE IS EQUALLY LIKELY TO PASS THROUGH THE BARRIER AS TO CROSS OVER THE WELL!

$\hookrightarrow$ THIS IS A DRAMATIC EXAMPLE OF <u>TUNNELING</u>!

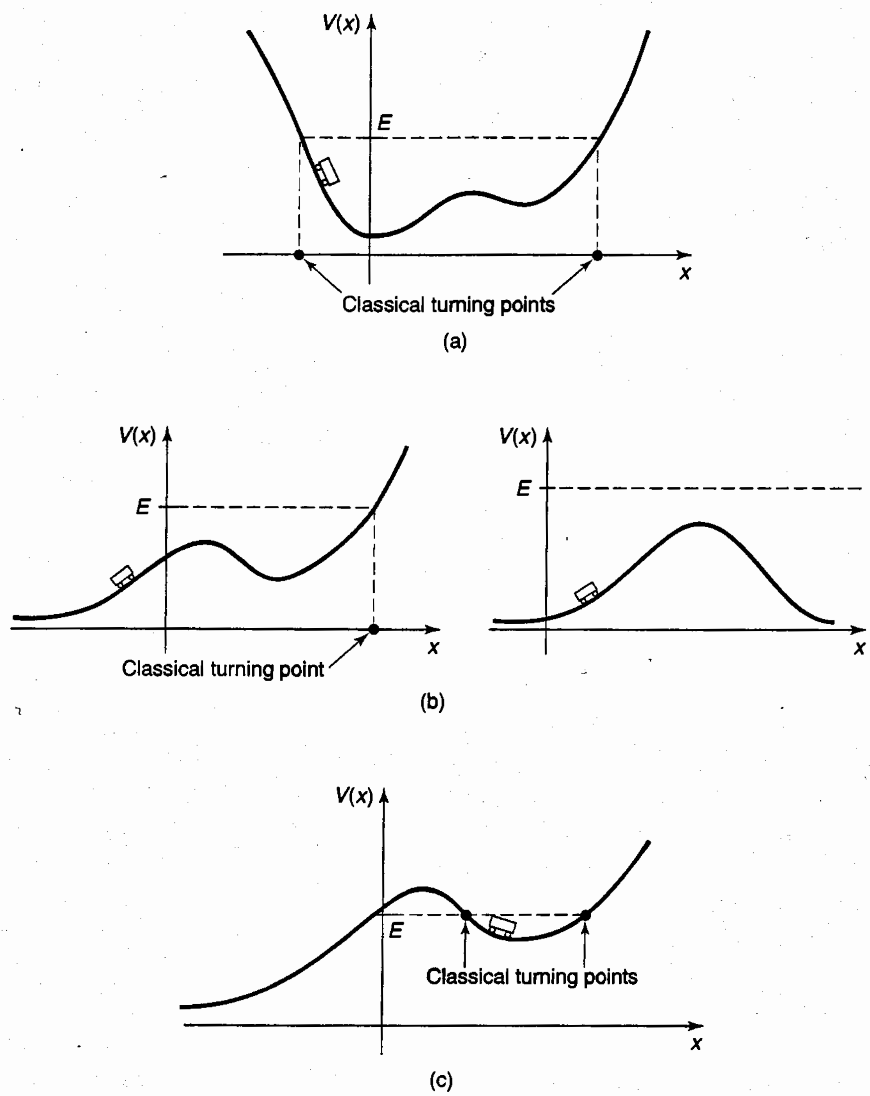


FIGURE 2.12: **(a) A bound state. (b) Scattering states. (c) A *classical* bound state, but a quantum scattering state.**