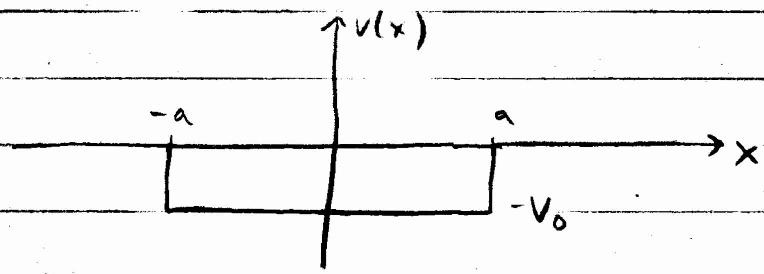
LECTURE #7

CONSIDER THE FINITE SQUARE WELL POTENTIAL:

$$V(x) = \begin{cases} -V_0 & \text{FOR } |x| \le a \\ 0 & \text{FOR } |x| > a \end{cases} \quad \text{WHERE } V_0 > 0$$



BOUND STATES ($E<0$):

$x<-a$: SCH. EQ. IS: $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E\psi$

$\Rightarrow \dfrac{d^2\psi}{dx^2} = K^2\psi$ WHERE $K = \dfrac{\sqrt{-2mE}}{\hbar}$

$\Rightarrow \psi(x) = Ae^{-Kx} + Be^{Kx}$

↳ THIS TERM "BLOWS UP" AS $x \to -\infty \Rightarrow A=0$

$\therefore \psi(x) = Be^{Kx}$, $x<-a$

$-a<x<a$: SCH EQ. IS: $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} - V_0\psi = E\psi$

$\Rightarrow \dfrac{d^2\psi}{dx^2} = -l^2\psi$ WHERE $l = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$

$\therefore \psi(x) = C\sin(lx) + D\cos(lx)$, $-a<x<a$

FOR $x>a$, IT SHOULD BE SELF-EVIDENT THAT $\psi(x) = Fe^{-Kx}$

THE SYMMETRY OF THE POTENTIAL IMPLIES THAT $\psi(x)$ MUST EITHER BE EVEN OR ODD. WE WILL CONSIDER THE EVEN SOLUTIONS HERE:

$$\text{i.e.,} \quad \psi(x) = \begin{cases} Fe^{-Kx}, & x > a \\ D\cos(\ell x), & 0 < x < a \\ \psi(-x), & x < 0 \end{cases}$$

ψ CONTINUOUS AT $x = a \Rightarrow \quad Fe^{-Ka} = D\cos(\ell a)$

$\frac{d\psi}{dx}$ CONTINUOUS AT $x = a \Rightarrow \quad -K Fe^{-Ka} = -\ell D\sin(\ell a)$

DIVIDING THESE 2 EQUATIONS $\Rightarrow \quad \ell\tan(\ell a) = K$

SINCE K AND ℓ ARE A FUNCTION OF E, THIS EQUATION REPRESENTS A FORMULA FOR THE ALLOWED ENERGIES.

LET $z = \ell a$, $\quad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$

NOTE: $K^2 + \ell^2 = \frac{2mV_0}{\hbar^2} = \frac{z_0^2}{a^2} \Rightarrow K^2a^2 + z^2 = z_0^2$
$$\Rightarrow Ka = \sqrt{z_0^2 - z^2}$$

$\therefore \quad \ell\tan(\ell a) = K \Rightarrow \ell a\tan(\ell a) = Ka \Rightarrow \tan(\ell a) = \frac{Ka}{\ell a}$

$$\therefore \quad \tan z = \sqrt{(z_0/z)^2 - 1}$$

↳ TRANSCENDENTAL EQUATION FOR z (AND HENCE FOR E)

↳ GRAPHICAL SOLUTION: $\boxed{\text{FIG. 2.18}}$ → FINITE NUMBER OF BOUND STATES

CONSIDER LIMITING CASES:

(1) WIDE, DEEP WELL $\Rightarrow z_0$ IS LARGE $\Rightarrow z_n \simeq \frac{n\pi}{2}$, n odd

$$z_n = \frac{a}{\hbar}\sqrt{2m(E_n+V_0)} \simeq \frac{n\pi}{2} \Rightarrow E_n + V_0 \simeq \frac{n^2\pi^2\hbar^2}{2m(2a)^2}, \quad n \text{ odd}$$

IN THE LIMIT OF LARGE z_0, THE FINITE WELL REDUCES TO THE INFINITE WELL (AS EXPECTED)

(2) SHALLOW, NARROW WELL $\Rightarrow z_0$ IS SMALL

WITH DECREASING z_0, THE NUMBER OF BOUND STATES DECREASES. ONCE $z_0 < \frac{\pi}{2}$ (WHERE THE LOWEST ODD STATE DISAPPEARS), ONLY ONE BOUND STATE REMAINS.

SCATTERING STATES ($E > 0$):

$x < -a$: $V(x) = 0 \Rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}$, $\quad k = \frac{\sqrt{2mE}}{\hbar}$

$-a < x < a$: $V(x) = -V_0 \Rightarrow \psi(x) = C\sin(lx) + D\cos(lx)$,

$$l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$x > a$: $\psi(x) = Fe^{ikx}$ (ONLY TRANSMITTED WAVE)

ψ CONTINUOUS AT $x = -a$: $Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la)$

$\frac{d\psi}{dx}$ CONTINUOUS AT $x = -a$: $ik[Ae^{-ika} - Be^{ika}] = l[C\cos(la) + D\sin(la)]$

ψ CONTINUOUS AT $x=a$: $C\sin(la) + D\cos(la) = Fe^{ika}$

$\frac{d\psi}{dx}$ CONTINUOUS AT $x=a$: $l[C\cos(la) - D\sin(la)] = ikFe^{ika}$

DIVIDE ALL 4 EQUATIONS BY A

$\Rightarrow$ 4 UNKNOWNS: $\frac{B}{A}, \frac{C}{A}, \frac{D}{A}, \frac{F}{A}$

SOLVING THIS 4x4 SERIES OF EQUATIONS IMPLIES:

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$

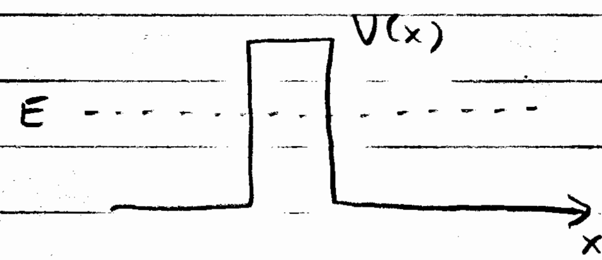
<u>NOTE</u>: $T=1$ (i.e, THE WELL IS TRANSPARENT) WHENEVER SINE IS ZERO.

$$\text{i.e,} \quad \frac{2a}{\hbar}\sqrt{2m(E_n+V_0)} = n\pi \Rightarrow E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

(THESE ARE THE ENERGIES OF THE INFINITE SQUARE WELL)

THE ENTIRE ENERGY DEPENDENCE OF T IS SHOWN IN FIG. 2.19

<u>NOTE</u>: THE RECTANGULAR BARRIER LEADS TO AN EXPONENTIAL DEPENDENCE OF T ON THE THICKNESS OF THE BARRIER $\Rightarrow$ <u>TUNNELING</u>



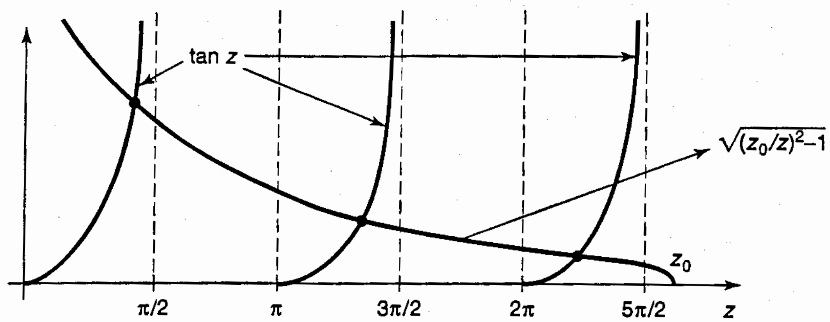


FIGURE 2.18: Graphical solution to Equation 2.156, for $z_0 = 8$ (*even* states).

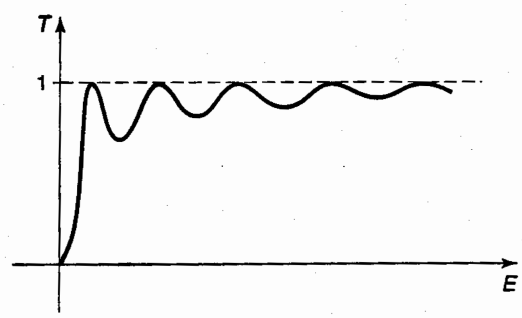


FIGURE 2.19: Transmission coefficient as a function of energy (Equation 2.169).