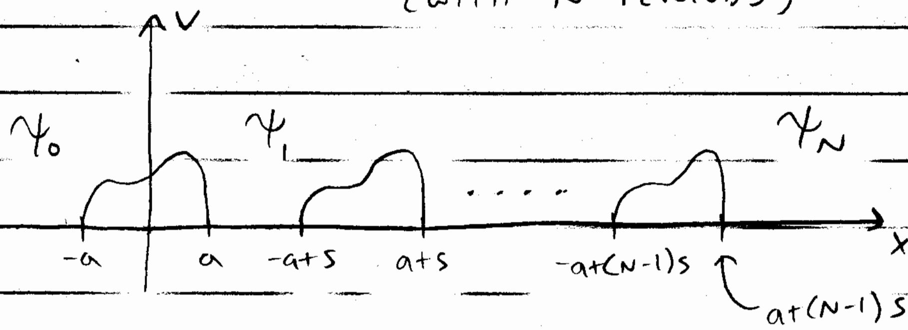
LECTURE #22

NOW CONSIDER A LOCALLY PERIODIC POTENTIAL:
(WITH N PERIODS)



OBJECTIVE: CONSTRUCT THE TRANSFER MATRIX FOR THE ENTIRE ARRAY GIVEN THE TRANSFER MATRIX FOR A SINGLE CELL.

THE WAVEFUNCTIONS BETWEEN THE CELLS ARE:

$$\psi_n(x) = A_n e^{ik(x-ns)} + B_n e^{-ik(x-ns)}$$

FOR $(n-1)s + a < x < ns - a$, $n = 0, 1, \dots, N$

$$\therefore \psi_0(x) = A_0 e^{ikx} + B_0 e^{-ikx}, \quad x < -a$$

$$\psi_1(x) = A_1 e^{-iks} e^{ikx} + B_1 e^{iks} e^{-ikx}, \quad a < x < -a+s$$

$\therefore$ USING OUR PREVIOUS TRANSFER MATRIX ANALYSIS,

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} w & z \\ z^* & w^* \end{pmatrix} \begin{pmatrix} A_1 e^{-iks} \\ B_1 e^{iks} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} w e^{-iks} & z e^{iks} \\ z^* e^{-iks} & w^* e^{iks} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

SIMILARLY, $$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} we^{-iks} & ze^{iks} \\ z^* e^{-iks} & w^* e^{iks} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$$

IN GENERAL, $$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = [P] \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix}$$

WHERE $$[P] = \begin{pmatrix} we^{-iks} & ze^{iks} \\ z^* e^{-iks} & w^* e^{iks} \end{pmatrix}$$

EMPLOYING THIS RELATIONSHIP RECURSIVELY FOR N PERIODS,

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = [P]^N \begin{pmatrix} A_N \\ B_N \end{pmatrix}$$

THE PROBLEM REDUCES TO CALCULATING THE N^{th} POWER OF A 2x2 MATRIX.

USING SEVERAL TRICKS FROM LINEAR ALGEBRA, IT CAN BE SHOWN THAT:

"XI"

$$[P]^N = [P]\, U_{N-1}(\xi) - I\, U_{N-2}(\xi)$$

WHERE $\xi = \frac{1}{2} \mathrm{Tr}\,[P]$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$U_N(\xi) = N^{th}$ CHEBYCHEV POLYNOMIAL OF THE SECOND KIND

$\therefore$ IN OUR CASE, $\xi = \frac{1}{2} \mathrm{Tr}\,[P] = \frac{1}{2}\left(we^{-iks} + w^* e^{iks} \right)$

$\therefore$ THE TRANSFER MATRIX FOR THE ENTIRE ARRAY OF N CELLS IS:

$$[M_N] = [P]^N \begin{pmatrix} e^{ikNs} & 0 \\ 0 & e^{-ikNs} \end{pmatrix}$$

$$= \begin{pmatrix} (we^{-iks}U_{N-1}(\xi) - U_{N-2}(\xi))e^{ikNs} & z\,U_{N-1}(\xi)e^{-ik(N-1)s} \\ z^* U_{N-1}(\xi)e^{ik(N-1)s} & (w^* e^{iks}U_{N-1}(\xi) - U_{N-2}(\xi))e^{-ikNs} \end{pmatrix}$$

$\therefore$ IF WE KNOW THE TRANSFER MATRIX FOR A SINGLE CELL (i.e., WE KNOW w AND z), WE CAN IMMEDIATELY WRITE DOWN THE TRANSFER MATRIX FOR N CELLS.

IN PARTICULAR, THE TRANSMISSION PROBABILITY FOR INCIDENCE FROM THE LEFT IS (ASSUMING $D=0$),

$$\begin{pmatrix} A \\ B \end{pmatrix} = [M]\begin{pmatrix} C \\ 0 \end{pmatrix} \Rightarrow A = M_{11}C \Rightarrow \left|\frac{C}{A}\right|^2 = \frac{1}{|M_{11}|^2}$$

$$\therefore T = \frac{1}{|M_{11}|^2}$$

RECALL: $|M_{11}|^2 - |M_{21}|^2 = 1$, $M_{21} = M_{12}^*$

$$\therefore |M_{11}|^2 = 1 + |M_{21}|^2 = 1 + |M_{12}|^2$$

$$\therefore T_N = \frac{1}{1+|M_{12}|^2} = \frac{1}{1+[|z|\,U_{N-1}(\xi)]^2}$$

NOTE: CHEBYSHEV POLYNOMIALS CAN BE WRITTEN AS:

$$U_N(\xi) = \frac{\sin(N+1)\gamma}{\sin\gamma} \quad \text{WHERE} \quad \gamma = \cos^{-1}\xi$$

$$\therefore \boxed{T_N = \left[1 + |z|^2 \left(\frac{\sin N\gamma}{\sin\gamma}\right)^2\right]^{-1}}$$

WHERE $\boxed{\gamma = \cos^{-1}\xi}$

AND $\xi = \frac{1}{2}\left(w e^{-iks} + w^* e^{iks}\right)$

$$= \frac{1}{2}\Big[\big(\mathrm{Re}(w) + i\,\mathrm{Im}(w)\big)(\cos ks - i\sin ks) + \big(\mathrm{Re}(w) - i\,\mathrm{Im}(w)\big)(\cos ks + i\sin ks)\Big]$$

$$= \frac{1}{2}\big[\mathrm{Re}(w)\cos ks + i\,\mathrm{Im}(w)\cos ks - i\,\mathrm{Re}(w)\sin ks + \mathrm{Im}(w)\sin ks + \mathrm{Re}(w)\cos ks - i\,\mathrm{Im}(w)\cos ks + i\,\mathrm{Re}(w)\sin ks + \mathrm{Im}(w)\sin ks\big]$$

$$\boxed{\xi = \mathrm{Re}(w)\cos ks + \mathrm{Im}(w)\sin ks}$$

THESE RESULTS CAN BE APPLIED TO SEVERAL PROBLEMS FOR EXAMPLE:

(1) TRANSMISSION THROUGH A PERIODIC δ POTENTIAL $\rightarrow$ FIG. 4

(2) ENERGY LEVELS FOR δ FUNCTIONS IN THE INFINITE SQUARE WELL $\rightarrow$ FIG. 7

(3) BY JOINING THE TAIL OF THE ARRAY TO ITS HEAD, THE KRONIG-PENNEY MODEL IS RECOVERED.

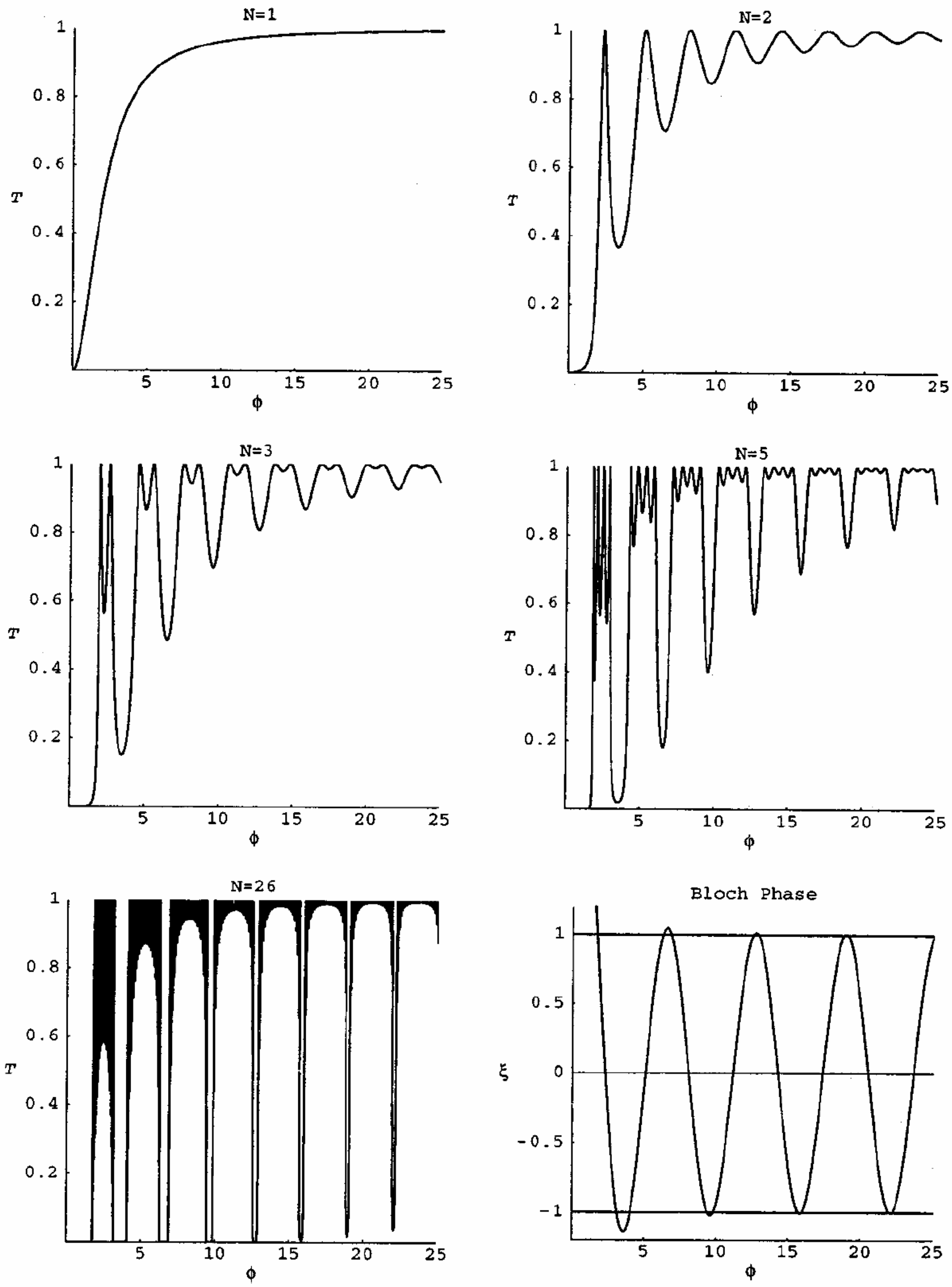


Fig. 4. Transmission coefficients for the periodic δ potential. The horizontal axis is $\phi = ks = s\sqrt{2mE}/\hbar$, and we used $mcs = 2\hbar^2$. In the last figure $\xi = \cos\,\gamma$, where γ is the Bloch phase.

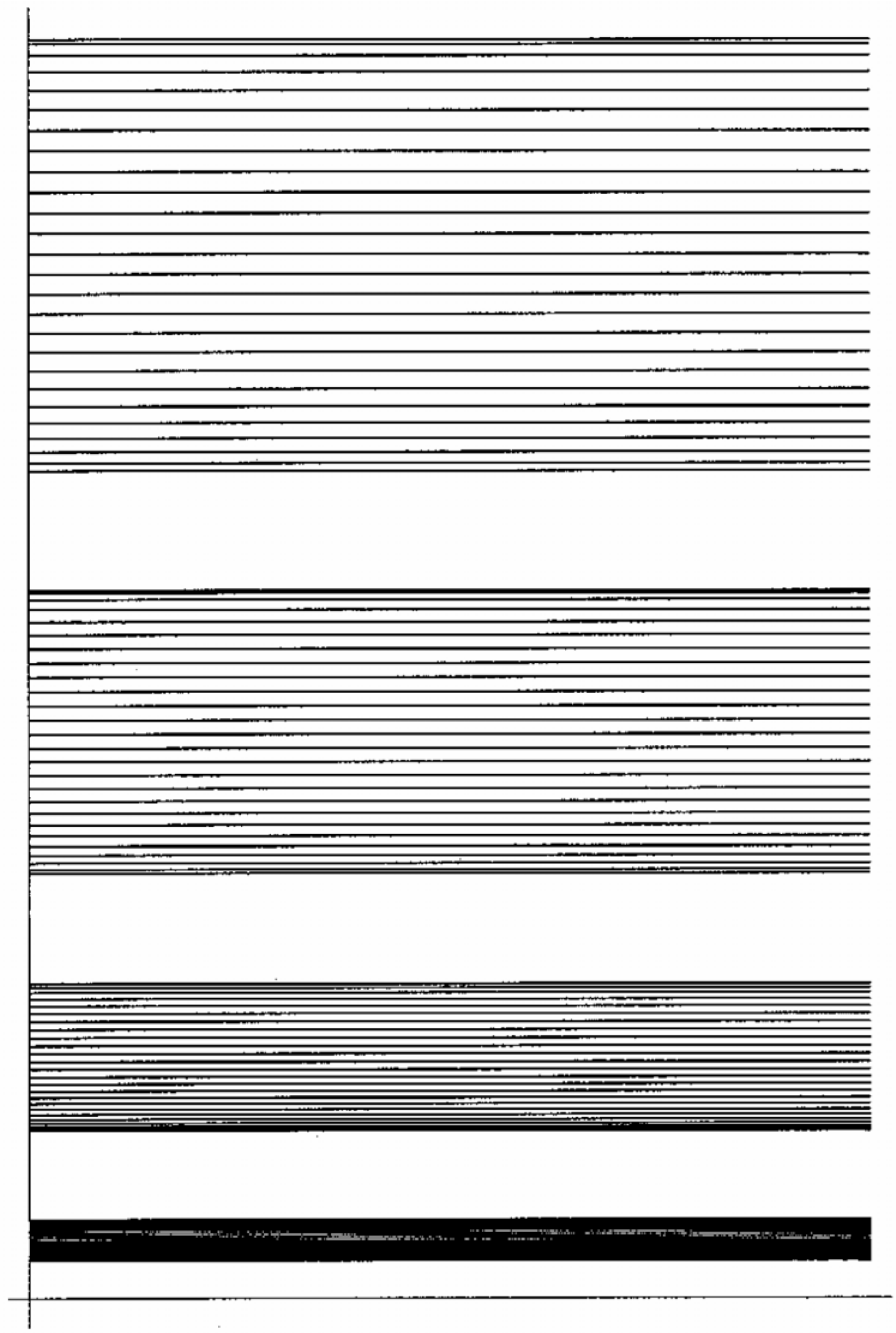


Fig. 7. Energy levels for 25 delta functions in the infinite square well.