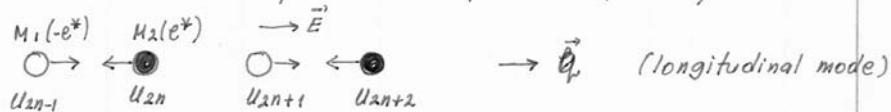


(c) POLAR OPTICAL PHONON SCATTERING - INTRAVALLEY

- In zinc-blende crystals there is always a small amount of ionic bonding because of the following reason: One atom has $Z > 4$ and the second one has $Z < 4$. Thus, there is a net electronic charge transfer from the atom with $Z > 4$ to the atom with $Z < 4$. For covalent bonds, in which the electrons are rattling around between different bonding orbitals, there is only a fractional charge transfer that is given by the EFFECTIVE CHARGE e^* (Callen effective charge).
- The small charge transfer leads to an effective dipole, which leads to finite ionic contribution to the dielectric function.
- The vibrations of the crystal lattice lead to oscillatory behavior of this dipole which, in turn, acts as a scattering potential.
- The coupled system of an electron plus longitudinal optical phonons is referred to as polaron. The electron-phonon interaction as applied to the polaron problem is described by the Fröhlich Hamiltonian.
- The goal of this section is two-fold:
 - (1) to find the Fröhlich Hamiltonian for the electron-longitudinal optical phonon interaction
 - (2) to calculate the scattering rate out of some initial state for this interaction

FRÖHLICH INTERACTION

- Since microscopic model is very difficult, a simpler approach is to consider a 1D diatomic lattice and calculate the ion contribution to the polarization function of the crystal.



- The equations of motion for this diatomic lattice, in the presence of electric field (longitudinal modes) are:

$$M_2 \frac{d^2 u_{2n}}{dt^2} = \beta (u_{2n+1} + u_{2n-1} - 2u_{2n}) + e^* E$$

$$M_1 \frac{d^2 u_{2n+1}}{dt^2} = \beta (u_{2n+2} + u_{2n} - 2u_{2n+1}) - e^* E$$

- In the case of long-wavelength phonons ($\vec{q} \rightarrow 0$), identical atoms will be displaced by the same amount, i.e.

$$u_{2n} = u_{2n+2} = u \quad \text{and} \quad u_{2n+1} = u_{2n-1} = w$$

This simplifies considerably the equations of motion to get:

$$\left\{ \begin{array}{l} M_2 \frac{d^2u}{dt^2} = \beta (2w - 2u) + e^* E \\ M_1 \frac{d^2w}{dt^2} = \beta (2u - 2w) - e^* E = 2\beta(u-w) - e^* E \end{array} \right.$$

or: $\left\{ \begin{array}{l} \frac{d^2u}{dt^2} = \frac{2\beta}{M_2} (w-u) + \frac{e^*}{M_2} E \\ \frac{d^2w}{dt^2} = \frac{2\beta}{M_1} (u-w) - \frac{e^*}{M_1} E \end{array} \right.$

Let $x = u - w$ describes the relative displacement of the two atoms within the unit cell. Then:

$$\frac{d^2u}{dt^2} - \frac{d^2w}{dt^2} = -\frac{2\beta}{M_2} (u-w) - \frac{2\beta}{M_1} (u-w) + e^* \left(\frac{1}{M_1} + \frac{1}{M_2} \right) E$$

$$\frac{d^2}{dt^2}(u-w) = -2\beta \left(\frac{1}{M_1} + \frac{1}{M_2} \right) (u-w) + e^* \left(\frac{1}{M_1} + \frac{1}{M_2} \right) E$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{2\beta}{M} x + \frac{e^*}{M} E} \quad (1) \quad \frac{1}{M} = \frac{1}{M_1} + \frac{1}{M_2} \rightarrow \text{reduced mass}$$

- The total dipole moment of the crystal is then given by:

$$e^* N_u X$$

and the polarization, which equals the dipole moment density, equals to:

$$P = \frac{e^* N_u X}{V}$$

(we have assumed that all atoms are displaced by the same amount, which gives identical x . N_u is the # of unit cells in the crystal).

By multiplying equation (1) by $e^* N_u$, we can recast it into an equation for the true dependence of the polarization associated with the ions in the presence of electric field E :

$$\frac{d^2P}{dt^2} + \frac{2\beta}{M} P = \frac{e^*}{M} \frac{e^* N_u}{V} E = \frac{e^{*2} N_u}{M N_u \Omega} E = \frac{e^{*2} E}{M \Omega}$$

where Ω is the volume of the unit cell. Now, assuming harmonic variation for the polarization, we get:

$$\text{let } \boxed{-\omega^2 P + \omega_{TO}^{-2} P = \frac{e^{*2} E}{M \Omega}} \Rightarrow P = \frac{e^{*2} E / (\bar{N} \Omega)}{\omega_{TO}^{-2} - \omega^2}$$

The displacement vector is :

$$\vec{D} = \epsilon_0 E + \vec{P} = \epsilon_0 E + \frac{e^{*2}/(\bar{M}\Omega)}{\omega_{r0}^2 - \omega^2} E = \epsilon(\omega)E$$

↑
high frequency dielectric constant (takes into account valence electron contribution)

Therefore, the frequency-dependent dielectric function $\epsilon(\omega)$, which incorporates the valence electrons contribution via ϵ_0 and the ionic contribution via \vec{P} , is given by:

$$\epsilon(\omega) = \epsilon_0 + \frac{e^{*2}/(\bar{M}\Omega)}{\omega_{r0}^2 - \omega^2} = \epsilon_0 + \frac{S}{\omega_{r0}^2 - \omega^2}$$

Setting $\omega=0$ gives us the static dielectric function, i.e.

$$\epsilon(0) = \epsilon_0 + \frac{S}{\omega_{r0}^2} \quad (3)$$

which gives for S :

$$S = \omega_{r0}^2 [\epsilon(0) - \epsilon_0] = \omega_{r0}^2 \epsilon(0) \epsilon_0 \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon(0)} \right)$$

- We now consider the case when there is no external field. The frequency $\omega = \omega_{LO}$, i.e. will be the natural frequency of the vibration of the longitudinal mode and the electric field that appears in (2) will be induced field, i.e.

$$-\omega_{LO}^2 P + \omega_{r0}^2 P = (\omega_{r0}^2 - \omega_{LO}^2) P = \frac{e^{*2} E_{ind}}{\bar{M}\Omega} = S E_{ind}$$

Since the material (crystal) contains no free charges, we must have :

$$\begin{aligned} \vec{D} \cdot \vec{D} &= q_D = (\epsilon_0 E_{ind} + P)_q = 0 \Rightarrow \epsilon_0 E_{ind} + P = 0 \Rightarrow \left\{ \begin{array}{l} P = -\epsilon_0 E_{ind} \\ E_{ind} = -\frac{P}{\epsilon_0} \end{array} \right. \\ \vec{D} \cdot \vec{D} &= q_D = q (\epsilon_0 E_{ind} + P) = 0 \end{aligned}$$

Substituting for E_{ind} gives:

$$(\omega_{r0}^2 - \omega_{LO}^2) P = S \left(-\frac{P}{\epsilon_0} \right) \Rightarrow S = \epsilon_0 (\omega_{LO}^2 - \omega_{r0}^2)$$

- (1) Substituting for S into the expression for the static dielectric function leads to:

$$\epsilon(0) = \epsilon_0 + \frac{\epsilon_0 (\omega_{LO}^2 - \omega_{r0}^2)}{\omega_{r0}^2} = \epsilon_0 \left[\frac{\omega_{r0}^2 + \omega_{LO}^2 - \omega_{r0}^2}{\omega_{r0}^2} \right] = \epsilon_0 \frac{\omega_{LO}^2}{\omega_{r0}^2}$$

or:

$$\frac{\epsilon(0)}{\epsilon_0} = \frac{\omega_{LO}^2}{\omega_{r0}^2} \quad (4) \quad \text{Lyddane - Sachs - Teller (LST) relation.}$$

- (2) Using LST relation back into equation (3) leads to the following expression for the effective charge:

$$S = \frac{e^*^2}{\bar{\mu} \Omega} = \omega_{L0}^2 \frac{\epsilon_{(0)}}{\epsilon(\omega)} \epsilon_{\infty} [\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon(0)}] = \omega_{L0}^2 \epsilon_{\infty}^2 [\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon(0)}]$$

Or:

$$e^*^2 = \bar{\mu} \Omega \omega_{L0}^2 \epsilon_{\infty}^2 \left(\frac{1}{\epsilon_{\infty}} - \frac{1}{\epsilon(0)} \right) \quad (5)$$

↑ static dielectric function
high-frequency

In the case of III-V compounds, $\bar{\mu} \Omega \omega_{L0}^2$ is approximately constant, independent of material, so a simple expression for the effective charge is:

$$\left(\frac{e^*}{e} \right)^2 \approx 3 \left[\frac{1}{K_{\infty}} - \frac{1}{K(0)} \right] \quad (\text{Ridley}).$$

- The next task is to calculate the Fröhlich Hamiltonian for the electron-phonon interaction. Again, we assume that there are no external fields applied. Then, the polarization P is due to the displacement of the atoms from their equilibrium positions and since there are no external charges $\nabla \cdot \vec{D} = 0$.

→ The polarization is given by:

$$P = \frac{e^* x(\vec{r})}{\Omega} = P(\vec{r}) \rightarrow \text{polarization due to the displacement of the two atoms in a unit cell located at } \vec{r}.$$

Expanding the displacement in the phonon coordinates gives:

$$P(\vec{r}) = \frac{e^*}{\Omega} \sum_q \sqrt{\frac{\hbar}{2 \bar{\mu} N_u w_q}} \vec{e}_q \left[\hat{a}_q e^{i\vec{q} \cdot \vec{r}} + \hat{a}_q^+ e^{-i\vec{q} \cdot \vec{r}} \right]$$

where \vec{e}_q is a unit vector in the direction of \vec{q} .

→ From the condition that $\nabla \cdot \vec{D} = 0$, we have that:

$$q_z [\epsilon_{\infty} \vec{E}_{\text{ind}} + \vec{P}] = 0 \quad (6)$$

Now let the electric field be derived from a potential, i.e.

$$\vec{E}_{\text{ind}} = -\nabla \Psi$$

and let the operator Ψ be represented by:

$$\Psi(\vec{r}) = \frac{1}{\sqrt{N_u}} \sum_q [\phi_q e^{i\vec{q} \cdot \vec{r}} + \phi_q^+ e^{-i\vec{q} \cdot \vec{r}}]$$

To determine ϕ_q 's, we substitute back into (6), to get:

$$-\frac{ie_0}{\sqrt{N_u}} \nabla \sum_q [\phi_q e^{i\vec{q} \cdot \vec{r}} + \phi_q^+ e^{-i\vec{q} \cdot \vec{r}}] + \frac{e^*}{\Omega} \sum_q \sqrt{\frac{\hbar}{2M N_u w_q}} \vec{e}_q [\hat{a}_q e^{i\vec{q} \cdot \vec{r}} + \hat{a}_q^+ e^{-i\vec{q} \cdot \vec{r}}] = 0$$

which gives:

$$\left\{ \begin{array}{l} -i\vec{q} \cdot \frac{e_0}{\sqrt{N_u}} \phi_q + \frac{e^*}{\Omega} \sqrt{\frac{\hbar}{2M N_u w_q}} \vec{e}_q \hat{a}_q = 0 \\ i\vec{q} \cdot \frac{e_0}{\sqrt{N_u}} \phi_q^+ + \frac{e^*}{\Omega} \sqrt{\frac{\hbar}{2M N_u w_q}} \vec{e}_q \hat{a}_q^+ = 0 \end{array} \right.$$

Since for longitudinal phonon modes $\vec{q} \parallel \vec{e}_q$, we finally have:

$$\left\{ \begin{array}{l} \phi_q = -i \frac{e^*}{e_0 q \Omega} \sqrt{\frac{\hbar}{2M w_q}} \hat{a}_q \\ \phi_q^+ = i \frac{e^*}{e_0 q \Omega} \sqrt{\frac{\hbar}{2M w_q}} \hat{a}_q^+ \end{array} \right.$$

The interaction energy between an electron located at \vec{r} and this field is given by:

$$\begin{aligned} H_{ep}(\vec{r}) &= -e\Phi(\vec{r}) = i \frac{ee^*}{e_0 \Omega} \sum_q \frac{1}{q} \sqrt{\frac{\hbar}{2M N_u w_q}} [\hat{a}_q e^{i\vec{q} \cdot \vec{r}} - \hat{a}_q^+ e^{-i\vec{q} \cdot \vec{r}}] \\ &= \sum_q \frac{iee^*}{e_0 \Omega q} \sqrt{\frac{\hbar}{2M N_u w_q}} [\hat{a}_q e^{i\vec{q} \cdot \vec{r}} - \hat{a}_q^+ e^{-i\vec{q} \cdot \vec{r}}] \end{aligned}$$

Therefore, the electronic part of the electron-Longitudinal polar optical phonons interaction is:

$$H_{qp}(\vec{r}) = \frac{iee^*}{e_0 \Omega q},$$

which for the coupling constant for $Inn' = \delta_{nn'}$ gives:

$$\begin{aligned} |C_{qp}|^2 &= \left(\frac{ee^*}{e_0 \Omega q} \right)^2 \delta(\vec{k} - \vec{k}' \pm \vec{q}) = \frac{e^2}{e_0^2 \Omega^2 q^2} \bar{M} \bar{\Omega} w_{lo}^2 \varepsilon_p^2 \left[\frac{1}{e_0} - \frac{1}{\varepsilon(0)} \right] \delta(\vec{k} - \vec{k}' \pm \vec{q}) \\ &= \frac{e^2 w_{lo}^2 \bar{M}}{\Omega q^2} \left[\frac{1}{e_0} - \frac{1}{\varepsilon(0)} \right] \delta(\vec{k} - \vec{k}' \pm \vec{q}) \end{aligned}$$

The matrix element for scattering between some initial state \vec{k} and some final state \vec{k}' is given by:

$$|M(\vec{k}, \vec{q})|^2 = \frac{\hbar}{2M N_u w_{lo}} \frac{e^2 w_{lo}^2 \bar{M}}{\Omega q^2} \left[\frac{1}{e_0} - \frac{1}{\varepsilon(0)} \right] \delta(\vec{k} - \vec{k}' \pm \vec{q}) \left[\frac{N_u}{N_{tot}} \right]^2 \text{ where } V = \Omega N_u \text{ is total volume.}$$

To summarize:

$$|H(\vec{k}, q)|^2 = \frac{\hbar e^2 \omega_{LO}}{2 V q^2} \left[\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon(0)} \right] \delta(\vec{k} - \vec{k}' \pm \vec{q}) \left[\frac{\sqrt{N_0}}{\sqrt{N_0 + 1}} \right]^2$$

SCATTERING RATE CALCULATION

Once we have the expression for $|H(\vec{k}, q)|^2$, we can use it for our calculation of the scattering rate where q_{min} and q_{max} are given by:

absorption:

$$q_{min}^{ab} = \hbar \left[\sqrt{1 + \frac{\hbar \omega_0}{E_K}} - 1 \right]$$

$$q_{max}^{ab} = \hbar \left[\sqrt{1 + \frac{\hbar \omega_0}{E_K}} + 1 \right]$$

emission:

$$q_{min}^{em} = \hbar \left[1 - \sqrt{1 - \frac{\hbar \omega_0}{E_K}} \right]$$

$$q_{max}^{em} = \hbar \left[1 + \sqrt{1 - \frac{\hbar \omega_0}{E_K}} \right]$$

Using these results we have:

$$\begin{aligned} \frac{1}{I(\vec{k})} &= \frac{m^* V}{2 \bar{U} \hbar^3 k} \int_{q_{min}^{ab}}^{q_{max}^{ab}} q |H(\vec{k}, q)|^2_{ab} dq + \frac{m^* V}{2 \bar{U} \hbar^3 k} \int_{q_{min}^{em}}^{q_{max}^{em}} q |H(\vec{k}, q)|^2_{em} dq \\ &= \frac{m^* V}{2 \bar{U} \hbar^3 k} \frac{\hbar e^2 \omega_{LO}}{2 V} \left[\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon(0)} \right] \left\{ N_0 \int_{q_{min}^{ab}}^{q_{max}^{ab}} \frac{1}{q} dq + (N_0 + 1) \int_{q_{min}^{em}}^{q_{max}^{em}} \frac{dq}{q} \right\} \\ &= \frac{m^* e^2 \omega_{LO}}{4 \bar{U} \hbar^2 k} \left[\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon(0)} \right] \left\{ N_0 \ln \left| \frac{\hbar \left[\sqrt{1 + \hbar \omega_0 / E_K} + 1 \right]}{\hbar \left[\sqrt{1 + \hbar \omega_0 / E_K} - 1 \right]} \right| + \right. \\ &\quad \left. + (N_0 + 1) \ln \left| \frac{\hbar \left[1 + \sqrt{1 - \hbar \omega_0 / E_K} \right]}{\hbar \left[1 - \sqrt{1 - \hbar \omega_0 / E_K} \right]} \right| \right\} \\ &= \frac{m^* e^2 \omega_{LO}}{4 \bar{U} \hbar^2 k} \left[\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon(0)} \right] \left\{ N_0 \ln \left| \frac{E_K + \sqrt{E_K + \hbar \omega_0}}{E_K - \sqrt{E_K + \hbar \omega_0}} \right| + (N_0 + 1) \ln \left| \frac{E_K + \sqrt{E_K - \hbar \omega_0}}{E_K - \sqrt{E_K - \hbar \omega_0}} \right| \right\} \end{aligned}$$

For parabolic bands, $E_K = \hbar^2 k^2 / 2m$, which gives $k = \sqrt{2m^* E_K / \hbar^2}$, i.e.

$$\frac{1}{I(\vec{k})} = \frac{\sqrt{m^*} e^2 \omega_{LO}}{4 \sqrt{2} \bar{U} \hbar \sqrt{E_K}} \left[\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon(0)} \right] \left\{ N_0 \ln \left| \frac{\sqrt{E_K + \hbar \omega_0} + E_K}{\sqrt{E_K + \hbar \omega_0} - E_K} \right| + (N_0 + 1) \ln \left| \frac{E_K + \sqrt{E_K - \hbar \omega_0}}{E_K - \sqrt{E_K - \hbar \omega_0}} \right| \right\}$$

ENERGY RELAXATION RATE CALCULATION

- Once we have obtained the expression for the scattering rate, we can immediately calculate the one for the energy relaxation rate since:

$$\begin{aligned}
 \frac{dE_K}{dt} &= \sum_{\vec{q}} \left[1 - \frac{E(\vec{k}')}{E(\vec{k})} \right] [S^{ab}(\vec{k}, \vec{q}) + S^{em}(\vec{k}, \vec{q})] \\
 &= \sum_{\vec{q}} [E_K - E_{K'}] [S^{ab}(\vec{k}, \vec{q}) + S^{em}(\vec{k}, \vec{q})] \\
 &= \sum_{\vec{q}} \left\{ [E_K - E_{K'} - \hbar\omega_0] S^{ab}(\vec{k}, \vec{q}) + [E_K - E_{K'} + \hbar\omega_0] S^{em}(\vec{k}, \vec{q}) \right\} \\
 &= \sum_{\vec{q}} \left\{ \hbar\omega_0 [S^{em}(\vec{k}, \vec{q}) - S^{ab}(\vec{k}, \vec{q})] = \hbar\omega_0 \left[\frac{1}{Z^{em}(\vec{k})} - \frac{1}{Z^{ab}(\vec{k})} \right] \right\} \\
 &= \hbar\omega_0 \frac{\sqrt{m^* e^2 \omega_0}}{4\sqrt{2} \bar{U} \sqrt{E_K}} \left[\frac{1}{\varepsilon_K} - \frac{1}{\varepsilon_{K'}} \right] \left\{ (N_0 + 1) \ln \left| \frac{E_K + \sqrt{E_K - \hbar\omega_0}}{E_K - \sqrt{E_K - \hbar\omega_0}} \right| - N_0 \ln \left| \frac{\sqrt{E_K + \hbar\omega_0} + E_K}{\sqrt{E_K + \hbar\omega_0} - E_K} \right| \right\} \\
 &= \frac{\sqrt{m^* e^2 \omega_0}}{4\sqrt{2} \bar{U} \sqrt{E_K}} \left[\frac{1}{\varepsilon_K} - \frac{1}{\varepsilon_{K'}} \right] \left\{ (N_0 + 1) \ln \left| \frac{E_K + \sqrt{E_K - \hbar\omega_0}}{E_K - \sqrt{E_K - \hbar\omega_0}} \right| - N_0 \ln \left| \frac{\sqrt{E_K + \hbar\omega_0} + E_K}{\sqrt{E_K + \hbar\omega_0} - E_K} \right| \right\}
 \end{aligned}$$

MOMENTUM RELAXATION RATE

- The momentum relaxation rate is more difficult to obtain. One must weight the absorption and the emission terms by the respective changes in the momentum. For example, an absorption of a phonon traveling at an angle θ to \vec{k} contributes a fractional increase of momentum of $(\vec{q}/k) \cos\theta$ in the direction of \vec{k} . The fractional increase in momentum for the emission process is $-(\vec{q}/k) \cos\theta$. To calculate the momentum relaxation rate, we must start with its general definition; and use the intermediate result we derived for $1/\tau(\vec{k})$:

$$\frac{1}{\tau_m(\vec{k})} = \frac{m^* V}{2\bar{U} \hbar^3 k} \int_{\vec{q}} \int_{\vec{k}'} \delta(\cos\theta) \left(1 - \frac{\vec{k} \cdot \vec{k}'}{k^2} \right) \delta\left(\frac{\vec{q}}{2k} \pm \cos\theta \mp \frac{m^* \omega}{\hbar k q}\right) |M(\vec{k}, \vec{q})|^2$$

Using the momentum conservation δ -function that appears in $|M(\vec{k}, \vec{q})|^2$, we have:

$$\begin{aligned}
 1 - \frac{\vec{k} \cdot \vec{k}'}{k^2} &= 1 - \frac{\vec{k} \cdot (\vec{k} \pm \vec{q})}{k^2} = 1 - \frac{k^2 \pm k q \cos\theta}{k^2} = \frac{k^2 - k^2 \mp k q \cos\theta}{k^2} \\
 1 - \frac{\vec{k} \cdot \vec{k}'}{k^2} &= \mp \frac{q}{k} \cos\theta \quad (\text{top sign} \rightarrow \text{absorption}, \text{bottom} \rightarrow \text{emission})
 \end{aligned}$$

Therefore:

$$\frac{1}{\tau_m(\vec{k})} = \frac{m^* V}{2\bar{U} \hbar^3 k} \int_{\vec{q}} \int_{\vec{k}'} \delta(\cos\theta) \left(\mp \frac{q}{k} \cos\theta \right) \delta\left(\frac{q}{2k} \pm \cos\theta \mp \frac{m^* \omega}{\hbar k q}\right) |M(\vec{k}, \vec{q})|^2$$

The δ -function will impose limits on the minimum and maximum values of wavevector q , to get:

$$\frac{1}{T_m(\vec{k})} = \frac{m^* V}{2\bar{\nu} \hbar^3 k} \int_{q_{\min}}^{q_{\max}} q \frac{q}{k} \left(\frac{q}{2k} + \frac{m^* \omega}{\hbar k q} \right) |H(\vec{k}, \vec{q})|^2 dq$$

$$= \frac{m^* V}{2\bar{\nu} \hbar^3 k} \frac{1}{2k^2} \int_{q_{\min}}^{q_{\max}} q^2 \left(q + \frac{m^* \omega}{\hbar k q} \cdot 2k \right) |H(\vec{k}, \vec{q})|^2 dq$$

$$\frac{1}{T_m(\vec{k})} = \frac{m^* V}{4\bar{\nu} \hbar^3 k^3} \int_{q_{\min}}^{q_{\max}} q^2 \left(q + \frac{2m^* \omega}{\hbar k q} \right) |H(\vec{k}, \vec{q})|^2 dq$$

General result for momentum relaxation time

Since the matrix element for polar optical phonon scattering $\sim 1/q^2$, to simplify the notation we get either for the absorption or the emission process:

$$\frac{1}{T_m(\vec{k})} = \frac{m^* V}{4\bar{\nu} \hbar^3 k^3} A \int_{q_{\min}}^{q_{\max}} \frac{q^2}{q^2} \left(q + \frac{2m^* \omega_0}{\hbar} \frac{1}{q} \right) dq$$

$$= \frac{m^* V A}{4\bar{\nu} \hbar^3 k^3} \left[\frac{1}{2} (q_{\max}^2 - q_{\min}^2) + \frac{2m^* \omega_0}{\hbar} \ln \left| \frac{q_{\max}}{q_{\min}} \right| \right]$$

$$= \frac{m^* V A}{2\bar{\nu} \hbar^3 k} \left[\frac{1}{4k^2} (q_{\max}^2 - q_{\min}^2) + \frac{2m^* \omega_0}{2\bar{\nu} / 2E_k / \hbar^2 k^2} \ln \left| \frac{q_{\max}}{q_{\min}} \right| \right]$$

$$= \frac{m^* V A}{2\bar{\nu} \hbar^3 k} \left[\frac{1}{4k^2} (q_{\max}^2 - q_{\min}^2) + \frac{\hbar \omega_0}{2E_k} \ln \left| \frac{q_{\max}}{q_{\min}} \right| \right]$$

$$= \frac{m^* V A}{2\bar{\nu} \hbar^3 k} \left[\frac{1}{4k^2} (q_{\max}^2 - q_{\min}^2) + \frac{\hbar \omega_0}{E_k} \ln \left| \frac{q_{\max}}{q_{\min}} \right|^{1/2} \right]$$

absorption: $q_{\min} = k \left[\sqrt{1 + \hbar \omega_0 / E_k} - 1 \right]$; $q_{\max} = k \left[\sqrt{1 + \frac{\hbar \omega_0}{E_k}} + 1 \right]$

$$q_{\max}^2 - q_{\min}^2 = (a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{q_{\max}}{q_{\min}} = \frac{a+b}{a-b} = \frac{(a+b)^2}{a^2 - b^2} \Rightarrow \left| \frac{q_{\max}}{q_{\min}} \right|^{1/2} = \frac{a+b}{\sqrt{a^2 - b^2}}$$

$$\left| \frac{q_{\max}}{q_{\min}} \right|^{1/2} = \frac{k \sqrt{1 + \hbar \omega_0 / E_k} + k}{k \sqrt{1 + \frac{\hbar \omega_0}{E_k}} - 1} = \frac{\sqrt{1 + \hbar \omega_0 / E_k} + 1}{\sqrt{\hbar \omega_0 / E_k}} = \left[\sqrt{\frac{E_k + \hbar \omega_0}{E_k}} + 1 \right] \sqrt{\frac{E_k}{\hbar \omega_0}}$$

$$= \sqrt{\frac{E_k}{\hbar \omega_0}} \cdot \left[\sqrt{1 + \frac{\hbar \omega_0}{E_k}} + 1 \right] = \sqrt{\frac{E_k}{\hbar \omega_0}} + \sqrt{1 + \frac{E_k}{\hbar \omega_0}} = \sinh^{-1} \sqrt{\frac{E_k}{\hbar \omega_0}}$$

$$\frac{1}{4K^2} (q_{\max}^2 - q_{\min}^2) = \frac{1}{4K^2} \left[1 + \frac{\hbar\omega_0}{EK} \right]^{1/2} \cdot k^2 = \sqrt{1 + \frac{\hbar\omega_0}{EK}}$$

Therefore:

$$\frac{1}{T_m(k)} = \frac{m^* V A}{2\bar{u}\hbar^3 k} \left[N_0 \sqrt{1 + \frac{\hbar\omega_0}{EK}} - N_0 \frac{\hbar\omega_0}{EK} \sinh^{-1} \left(\frac{EK}{\hbar\omega_0} \right)^{1/2} \right]$$

In deriving the above expression, we have used that:

$$\sinh^{-1}(x) = \ln [x + \sqrt{1+x^2}]$$

emission : $q_{\min} = -k\sqrt{1 - \frac{\hbar\omega_0}{EK}} + k = k - k\sqrt{1 - \frac{\hbar\omega_0}{EK}}$

$$q_{\max} = k + k\sqrt{1 - \frac{\hbar\omega_0}{EK}}$$

$$\frac{1}{4K^2} (q_{\max}^2 - q_{\min}^2) = \frac{1}{4K^2} \cdot 4ab = \frac{1}{K^2} k \cdot k \sqrt{1 - \frac{\hbar\omega_0}{EK}} = \sqrt{1 - \frac{\hbar\omega_0}{EK}}$$

$$\begin{aligned} \left| \frac{q_{\max}}{q_{\min}} \right|^{1/2} &= \sqrt{\frac{(a+b)^2}{|a^2 - b^2|}} = \sqrt{\frac{k^2 (1 + \sqrt{1 - \frac{\hbar\omega_0}{EK}})^2}{k^2 \left| 1 - \frac{\hbar\omega_0}{EK} + 1 \right|}} = \sqrt{\frac{EK}{\hbar\omega_0} \left(1 + \sqrt{1 - \frac{\hbar\omega_0}{EK}} \right)^2} \\ &= \sqrt{\frac{EK}{\hbar\omega_0}} \left(1 + \sqrt{1 - \frac{\hbar\omega_0}{EK}} \right) = \sqrt{\frac{EK}{\hbar\omega_0}} + \sqrt{\frac{EK}{\hbar\omega_0} - 1} \\ &= \sqrt{\frac{EK}{\hbar\omega_0} - 1} + \sqrt{\left(\frac{EK}{\hbar\omega_0} - 1 \right) + 1} \end{aligned}$$

Hence:

$$\frac{1}{T_m(k)} = \frac{m^* V A}{2\bar{u}\hbar^3 k} (N_0 + 1) \left[\sqrt{1 - \frac{\hbar\omega_0}{EK}} + \frac{\hbar\omega_0}{EK} \sinh^{-1} \sqrt{\frac{EK}{\hbar\omega_0} - 1} \right]$$

Substituting for $A = \frac{\hbar e^2 \omega_0}{2V\varepsilon_p}$ and combining the absorption and the emission process gives the following expression for the momentum relaxation rate:

$$\frac{1}{T_m(k)} = \frac{m^* \omega_0^2 \left(\frac{1}{E_0} - \frac{1}{E_{\text{rel}}} \right)}{4\bar{u}\hbar^2 \sqrt{2m^* E_K/\hbar^2}} \cdot \left\{ N_0 \sqrt{1 + \frac{\hbar\omega_0}{EK}} - N_0 \frac{\hbar\omega_0}{EK} \sinh^{-1} \left(\frac{EK}{\hbar\omega_0} \right)^{1/2} \right. \\ \left. + (N_0 + 1) \sqrt{1 - \frac{\hbar\omega_0}{EK}} + (N_0 + 1) \frac{\hbar\omega_0}{EK} \sinh^{-1} \left(\frac{EK}{\hbar\omega_0} - 1 \right)^{1/2} \right\}$$