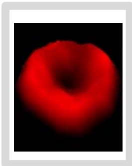

Chapter 3 – Geometrical Optics

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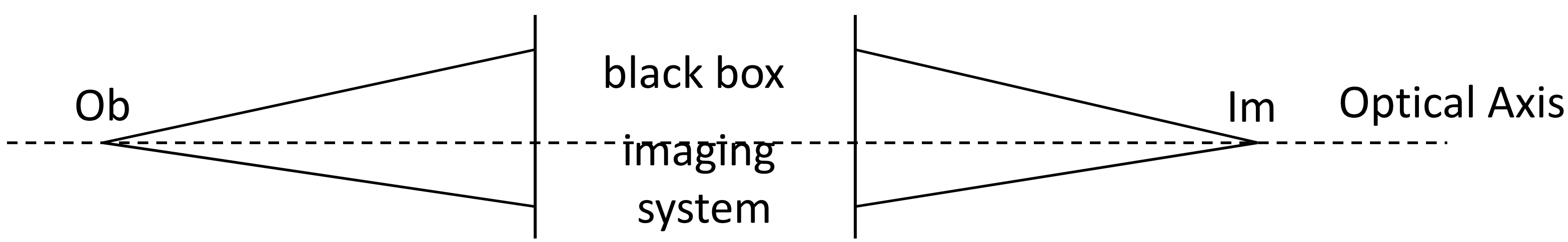
Objectives

- Introduction to geometrical optics and Fourier optics – precedes Microscopy

3.1 Geometrical Optics

- If the objects encountered by light are large compared to wavelength, the equations of propagation can be greatly simplified ($\lambda \rightarrow 0$)
- i.e. the wave-phenomena (scattering, interference, etc) are neglected
- In homogeneous media, light travels in straight lines ≡ rays
- G.O. deals with ray propagation trough optical media (eg. Imaging systems)

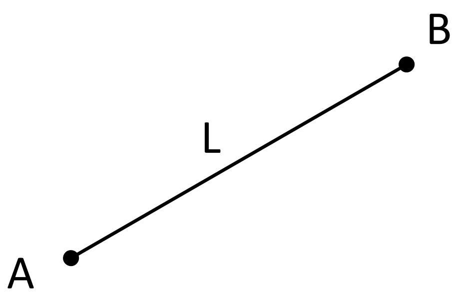


3.1 Geometrical Optics

- G.O. predicts image location trough complicated systems; accuracy is fairly good
- Nowadays there are software programs that can run "ray propagation" trough arbitrary materials
- So, what are the laws of G.O.?

3.2 Fermat's principle

a) n = constant

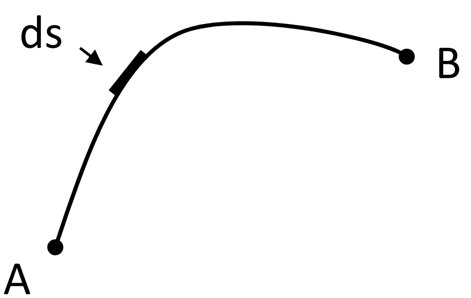


$$v = \frac{c}{n}$$

Time: $t_{AB} = \frac{L}{v} = \frac{1}{c} nL$

→ straight line

b) n = n($\bar{r}$) = function of position



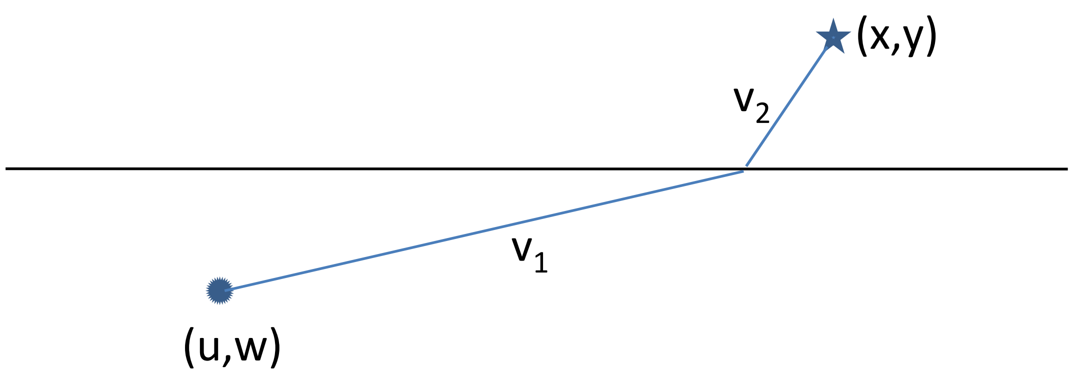
$$v(r) = \frac{c}{n(r)}$$

$$dt = \frac{ds}{v_B} = \frac{1}{c} n(s) ds$$

→ $$t_{AB} = \frac{1}{c} \int_A n(s) ds \qquad (3.1)$$

3.2 Fermat's principle

- Fermat's Principle is reminiscent of the following problem that you might have seen in highschool:
  - Someone is drowning in the Ocean at point (x,y) The lifeguard at point (u,w) can travel across the beach at speed v_1 and in the water at speed v_2. What is his best possible path?

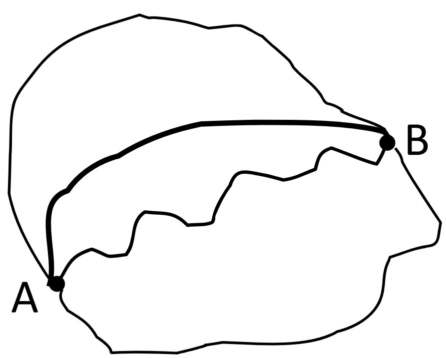


3.2 Fermat's principle

- Definition:

$S = ct = \int n(s)ds \equiv$ optical path length (3.2)

- How can we predict ray bending (eg. mirage)?
- Fermat's Principle:
  - Light connects any two points by a path of minimum time (the least time principle)

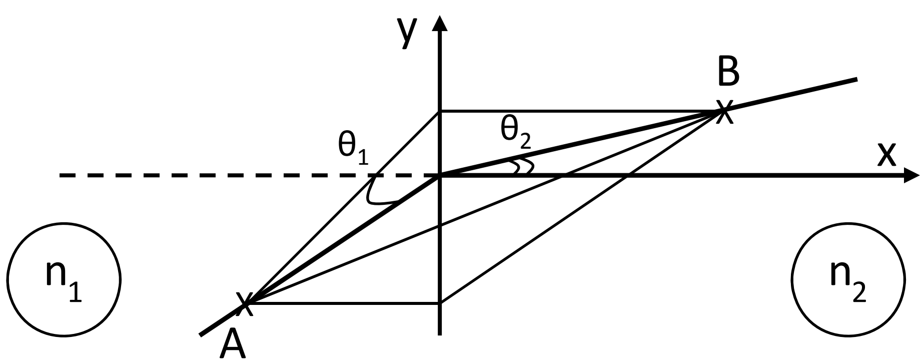


$$\delta\left[\int_A^B n(S)dS\right] = 0 \quad (3.3)$$

  - If n=constant in space, AB=line, of course

3.3 Snell's Law

- Consider an interface between 2 media:



- The rays are "bent" such that: (3.4)

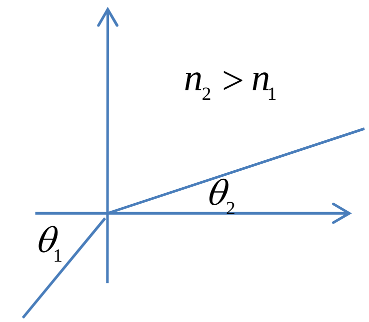
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- Snell's law (3.4) can be easily derived from Fermat's principle, by minimizing:

$$S = n_1 |AO| + n_2 |OB| = \text{total path-length}$$

- Take it as an exercise

3.3 Snell's Law



- Consequences of Snell's Law:

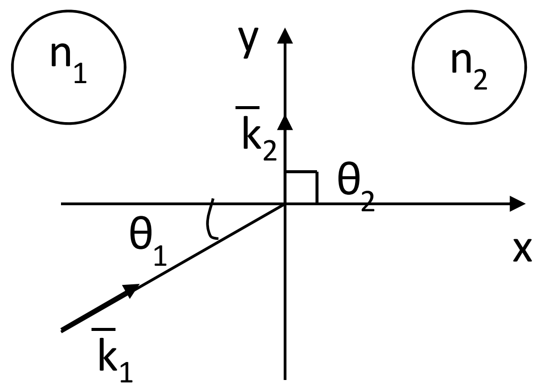
a) If $n_2 > n_1$ → $\Theta_2 < \Theta_1$ (ray gets closer to normal)

b) If $n_2 < n_1$ → quite interesting!

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_1\right] \qquad (3.5)$$

→ Ray gets away from normal

$n_2 < n_1$



$$\bar{k} = \frac{2\pi}{\lambda}\hat{k}$$

- So, if $\left[\frac{n_1}{n_2}\sin\theta_1\right] = 1 \Rightarrow \boxed{\theta_2 = \pi/2}$ NO TRANSMISSION

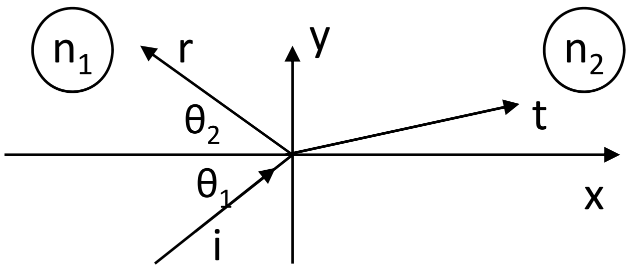
3.3 Snell's Law

- The angle of incidence θ_c for which

$$n_1 \sin\theta_c = n_2 \qquad (3.6)$$

  is called critical angle

- This is total internal reflection



c) law of reflection

$n_2 = -n_1$ Snell's law is:

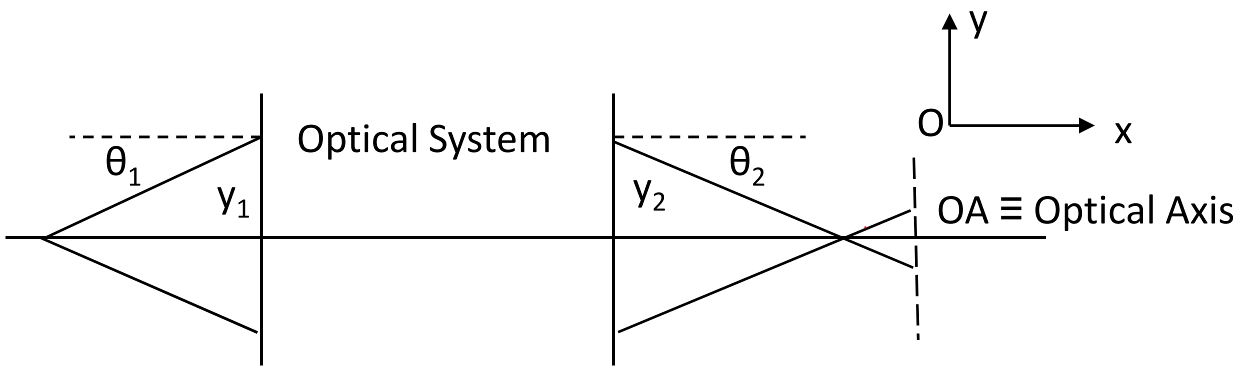
$$\rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\rightarrow \quad \theta_1 = -\theta_2 \text{ (reflection law)} \qquad (3.7)$$

- Energy conservation: $P_t + P_r = P_i$

3.4 Propagation Matrices in G.O

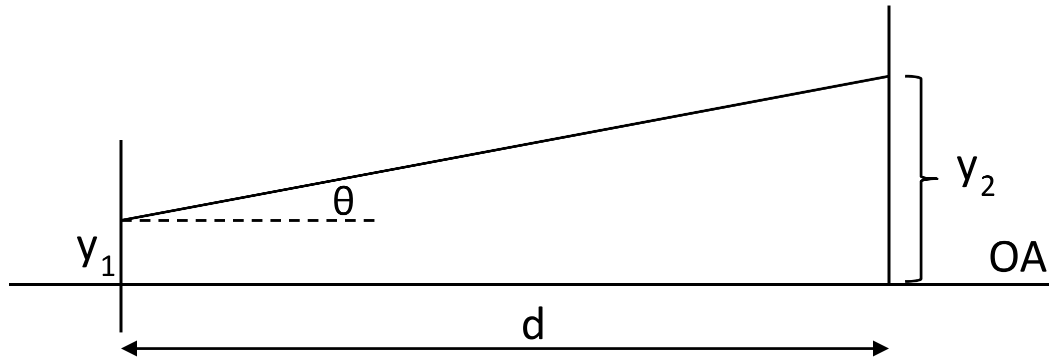
- Efficient way of propagating rays through optical systems



- Any given ray is completely determined at a certain plane by the angle with OA, Θ_1, and height w.r.t OA, y_1 ≡
- Let's propagate (y_1, Θ_1), assume small angles <u>Gaussian approximation</u>

3.4 Propagation Matrices in G.O

a) Translation



$$\begin{cases} \theta_2 = \theta_1 \\ y_2 = y_1 + d\tan\theta_1 \end{cases}$$

Small angles:

$$\begin{cases} y_2 = y_1 + d\theta_1 \\ \theta_2 = 0y_1 + 1\theta_1 \end{cases} \qquad (3.8)$$

3.4 Propagation Matrices in G.O

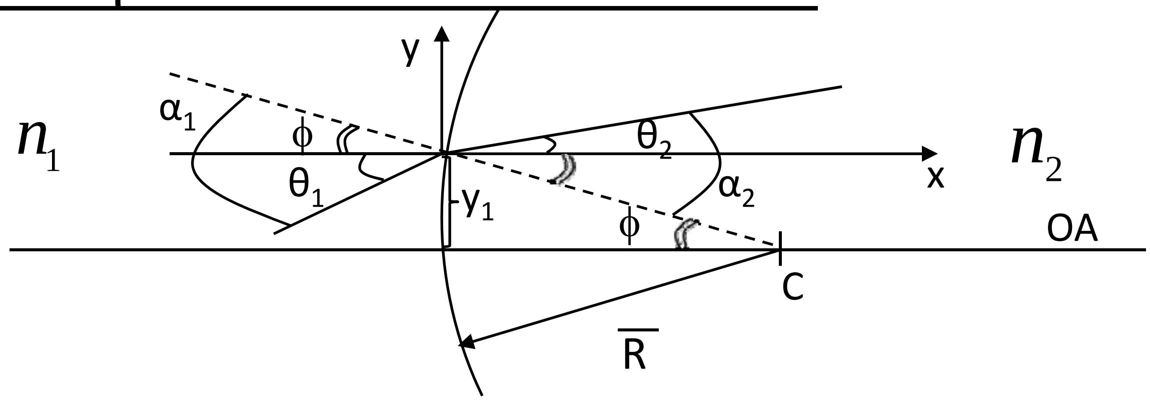
a) Translation

- We can re-write in compact form:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \qquad (3.9)$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dielectric interface



- Snell's law: $n_1\alpha_1 = n_2\alpha_2$
- Geometry: $\begin{cases} \alpha_1 = \theta_1 + \phi \\ \alpha_2 = \theta_2 + \phi \\ \phi = \dfrac{y_1}{R} = \dfrac{y_2}{R} \end{cases}$ (3.10)

$$\rightarrow n_1\theta_1 + \frac{n_1}{R} y_1 = n_2\theta_2 + \frac{n_2}{R} y_2 \mid \cdot \frac{1}{n_2}$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- So: $$\begin{cases} y_2 = y_1 + 0 \cdot \theta_1 \\ \theta_2 = (\frac{n_1}{n_2} - 1)\frac{y_1}{R} + \frac{n_1}{n_2}\theta_1 \end{cases}$$

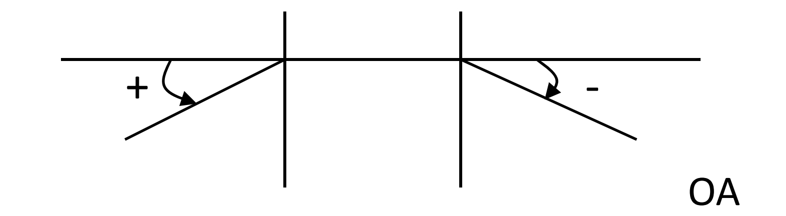
→ $$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \qquad (3.11)$$

3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- Important: To avoid confusion between Θ and – Θ angles, use "sign convention"

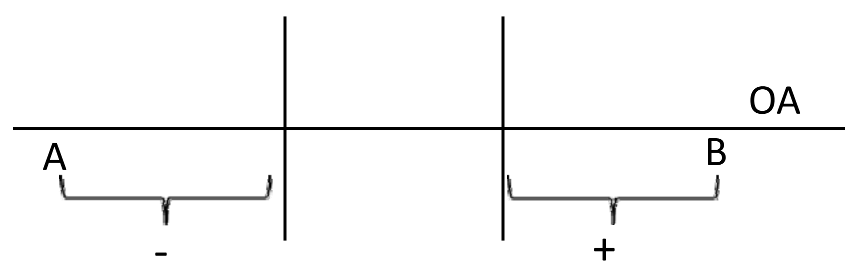
1. angle convention



- Counter clock-wise = positive

2. distance convention

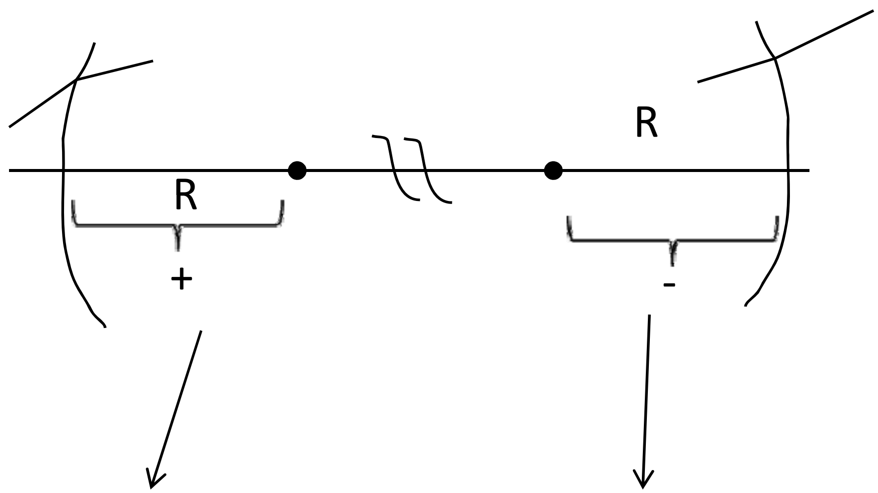
Left → negative
Right → positive



3.4 Propagation Matrices in G.O

b) Refraction-spherical dieletric interface

- Example:

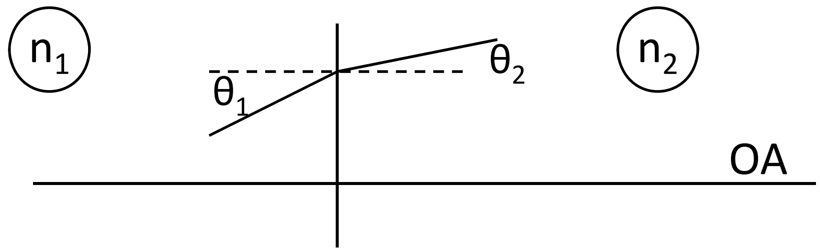


We found $\begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix}$

Same +/- convention applies to spherical mirrors. Without sign convention, it's easy to get the wrong numbers.

3.4 Propagation Matrices in G.O

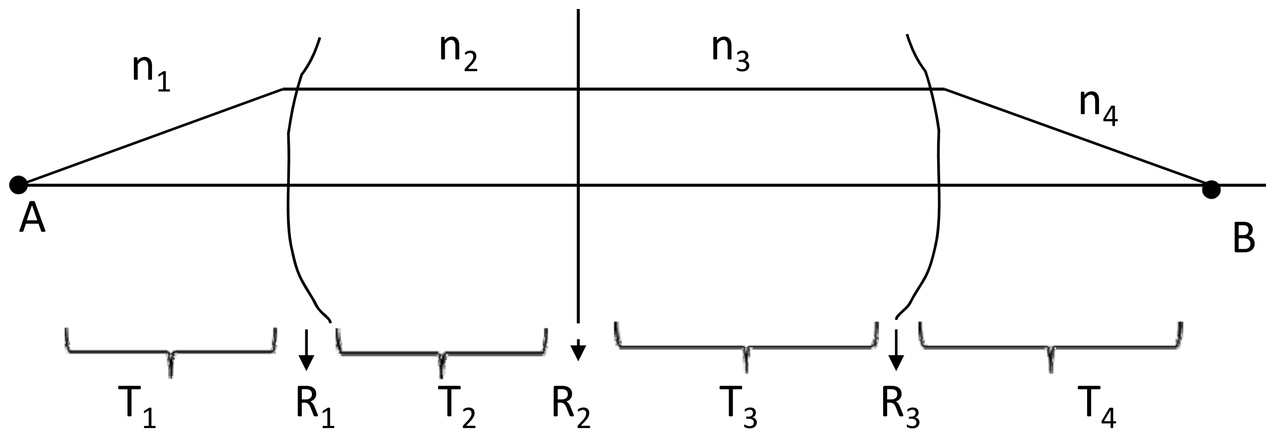
c) Dielectric interface – particular case of R→ ∞



$$\lim_{R\to\infty}\begin{pmatrix} 1 & 0 \\ \dfrac{n_1-n_2}{n_2 R} & \dfrac{n_1}{n_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{pmatrix} \qquad (3.12)$$

3.4 Propagation Matrices in G.O

- The nice thing is that cascading multiple optical components reduces to multiplying matrices (linear systems)
- Example:



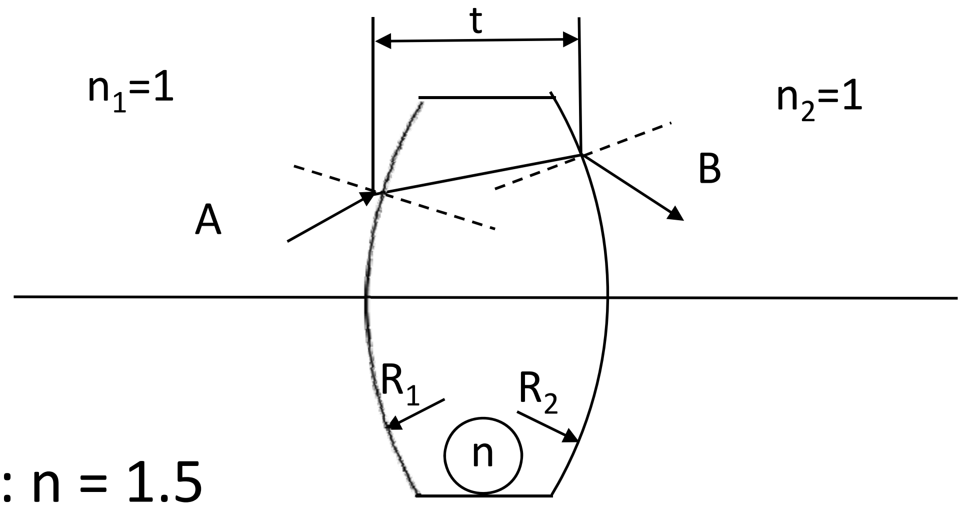
$$\begin{pmatrix} y_B \\ \theta_B \end{pmatrix} = T_4 \cdot R_3 \cdot T_3 \cdot R_2 \cdot T_2 \cdot R_1 \cdot T_1 \cdot \begin{pmatrix} y_A \\ \theta_A \end{pmatrix} \quad (3.13)$$

- Note the reverse order multiplication (chronological order)

3.4 Propagation Matrices in G.O

- Note the reverse order multiplication (chronological order)
- T = Translation matrix = $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$
- R=refraction matrix = $\begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{pmatrix}$

3.5 The thick Lens



- Typical glass: n = 1.5
- Basic optical component: typically - 2 spherical surfaces

$$\begin{pmatrix} y_B \\ \theta_B \end{pmatrix} = R_B \cdot T_t \cdot R_A . \begin{pmatrix} y_A \\ \theta_A \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ \dfrac{n-1}{R_2} & n \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \dfrac{1-n}{nR_1} & \dfrac{1}{n} \end{pmatrix}}_{M} . \begin{pmatrix} y_A \\ \theta_A \end{pmatrix}$$

3.5 The thick Lens

- After some algebra:

$$M = \begin{pmatrix} 1 - C_1 \frac{t}{R_1} & \frac{t}{n} \\ -(C_1 + C_2 - C_1 C_2 \frac{t}{n}) & 1 - C_2 \frac{t}{nR_2} \end{pmatrix} \quad (3.14)$$

- In general $C = \frac{n_2 - n_1}{R} \equiv$ convergence of spherical surface
- $R_1 > 0$, $R_2 < 0$ → $C_1 > 0$ & $C_2 > 0$ → <u>convergent</u>
- Note [C] = m^{-1} = dioptries

3.5 The thick Lens

- Definition: $$\frac{1}{f} = C_1 + C_2 - C_1 C_2 \frac{t}{n} \qquad (3.15)$$
- f is the focal distance of lens
- Eq (3.15) is the "lens makers equation"

3.6 Cardinal points

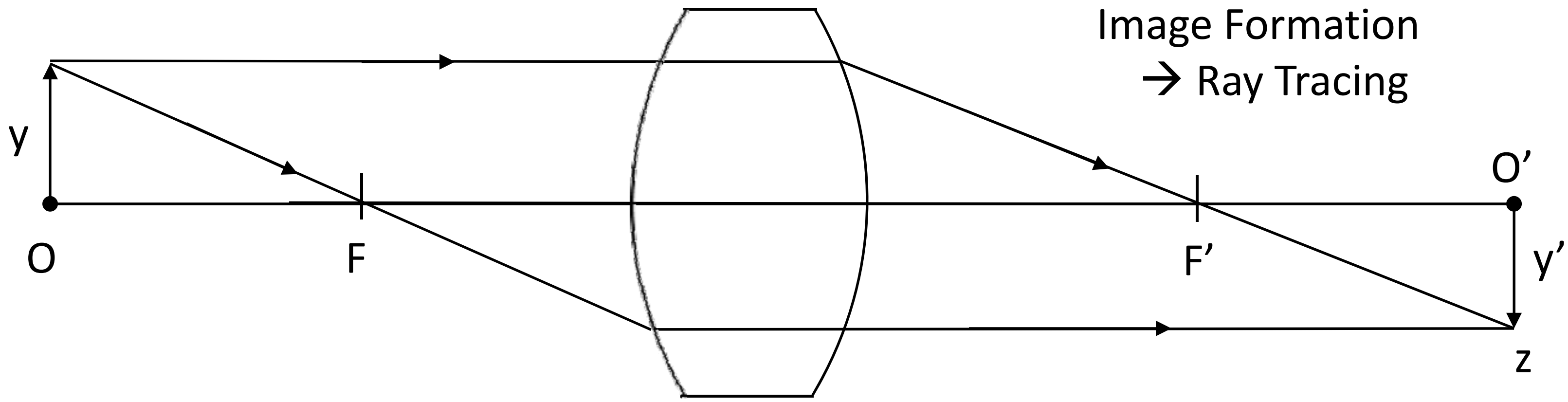


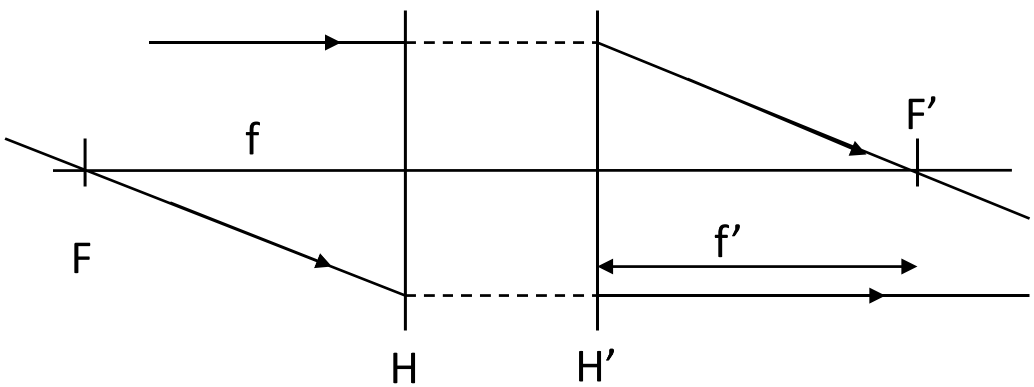
Image Formation
→ Ray Tracing

- O = object; O'=image ; O-O'=conjugate points
- F' = focal point image (image of objects from -∞)
- F = focal point object
- Transverse magnification:

$$M = \frac{y'}{y} \qquad (3.16)$$

3.6 Cardinal points

- Definition: principal planes are the conjugate planes for which M = 1



H, H' = principal planes

f, f' = focal distances

! f, f' measured from H

3.7 Thin lens

- Particular use: t → 0
- Transfer matrix for thin lens:

$$\lim_{t\to 0}\begin{pmatrix} 1-C_1\dfrac{t}{R_1} & \dfrac{t}{n} \\ -(C_1+C_2-C_1C_2\dfrac{t}{n}) & 1-C_2\dfrac{t}{nR_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(C_1+C_2) & 1 \end{pmatrix}$$

- Since $\dfrac{1}{f} = C_1 + C_2 = (n-1)(\dfrac{1}{R_1} - \dfrac{1}{R_2})$
- (Note $R_1 > 0$, $R_2 < 0$)

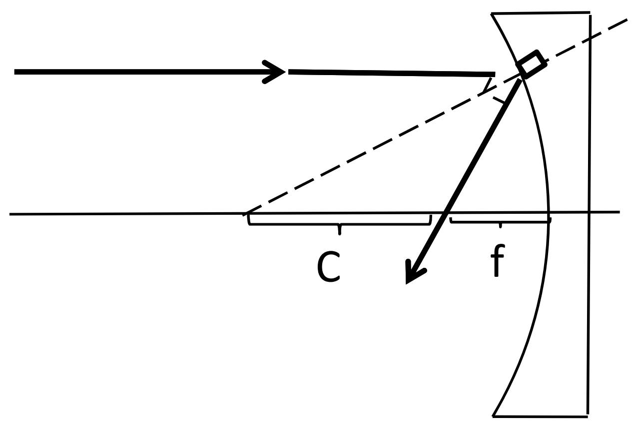
→ $M_{\text{thin lens}} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}$ (3.17)

3.7 Thin lens

- Remember other matrices:
  - Translation: $$T = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{3.18}$$
  - Refraction-spherical surface: $$R = \begin{pmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{pmatrix} \tag{3.19}$$
  - Spherical mirror: (f = R/2) $$M = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \tag{3.20}$$

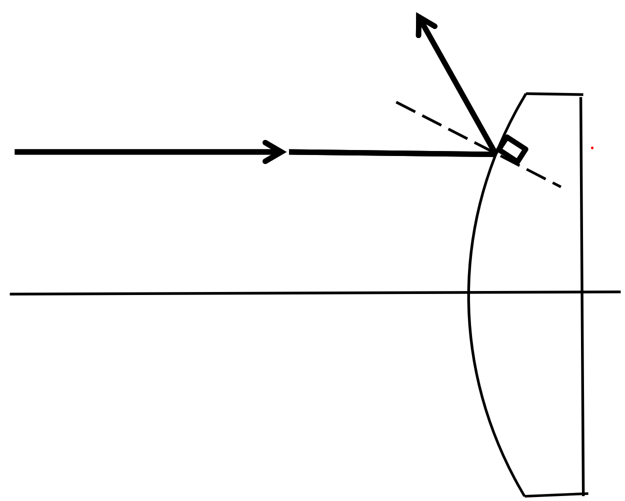
3.7 Spherical Mirrors

Convergent

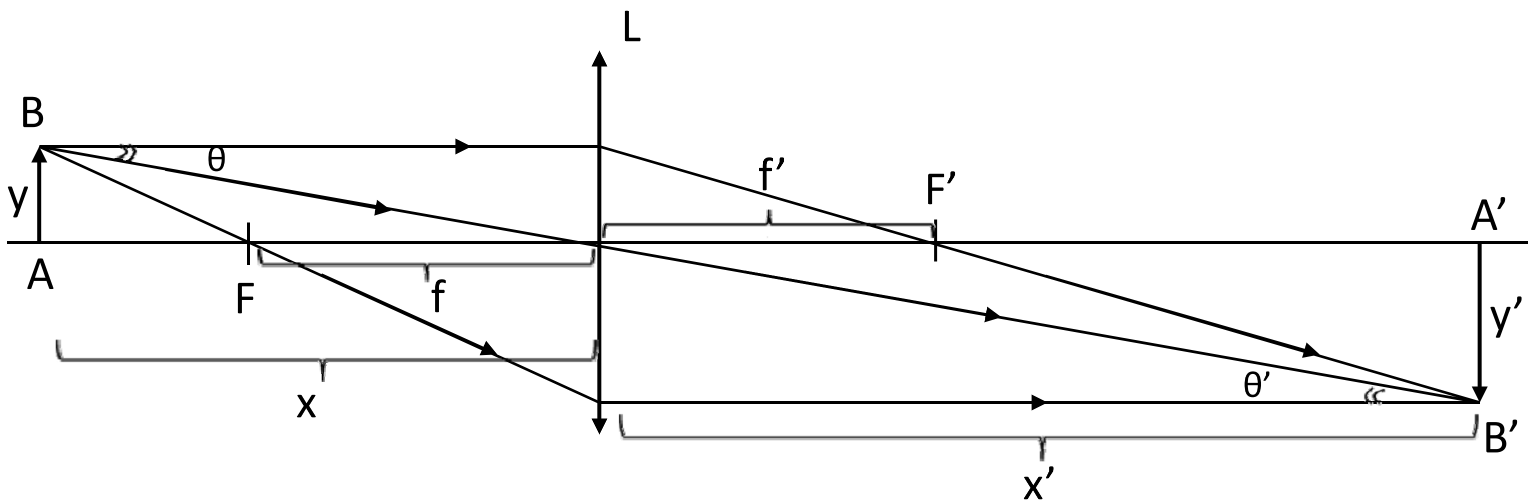


$$f \simeq \frac{R}{2}$$

Divergent



3.8 Ray Tracing – thin lenses



-  = convergent lens; $f > 0$
-  = divergent lens; $f < 0$

3.8 Ray Tracing – thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = T_{x'} M_f T_x \begin{pmatrix} y \\ \theta \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix} =$$

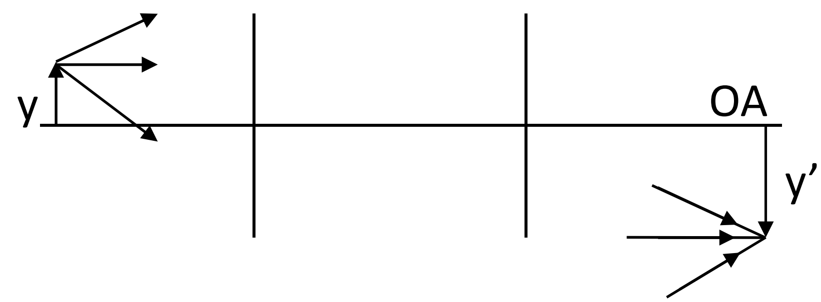
$$= \begin{pmatrix} 1-\frac{x'}{f} & x+x'-\frac{xx'}{f} \\ -\frac{1}{f} & 1-\frac{x}{f} \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix} \quad (3.21)$$

3.8 Ray Tracing – thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$; y’ can be found as:

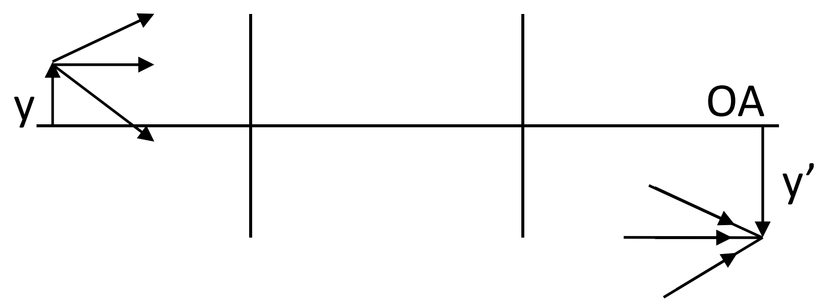
$$\boxed{y' = Ay + B\theta} \qquad (3.22)$$

- Condition for conjugate planes:



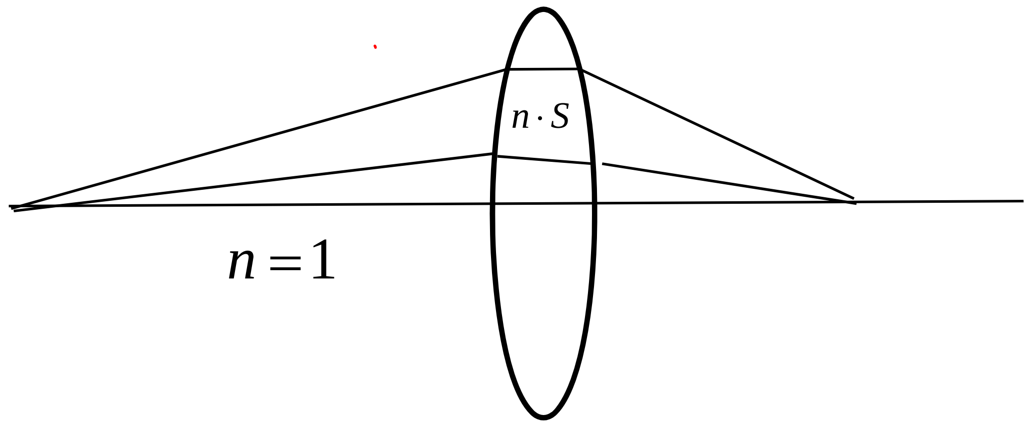
- For conjugate planes, y’ should be independent of angle Θ
  → $\boxed{B = 0}$
- i.e. stigmatism condition (points are imaged into points)
- We neglect geometric/chromatic aberrations

Quiz



Explain how Fermat’s principle works here.

Solution



$$S = \int n(s)ds$$

Because all of the rays leaving a given point converge again in the image, we know from Fermat's principle that their paths must all take the same amount of time. Another way to say this is that all the paths have the same optical path length. This is because those paths that travel further in the air, have a "shorter" distance to travel in the more time-expensive glass. If the optical path lengths were not the same, the image would not be in focus because rays from a single point would be mapped to several points.

3.8 Ray Tracing – thin lenses

- So, B = 0 → $x + x' - \frac{xx'}{f} = 0$

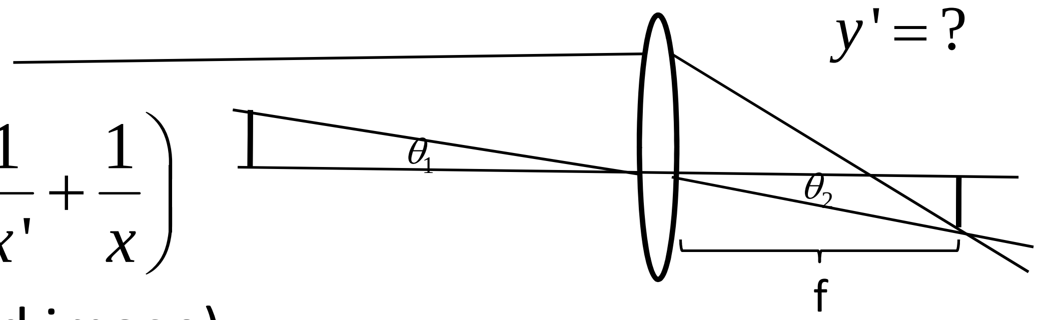
  → $\boxed{\frac{1}{x'} + \frac{1}{x} = \frac{1}{f}}$ (3.23)

- Eq above is the conjugate points equation (thin lens)
- Eq 3.22 becomes: y'= yA

$$\rightarrow M = A = 1 - \frac{x'}{f} = \text{Transverse magnification} \qquad (3.24)$$

3.8 Ray Tracing – thin lenses

- Use Eq 3.23:



$$M = 1 - \frac{x'}{f} = 1 - x'\left(\frac{1}{x'} + \frac{1}{x}\right)$$
$$= \frac{x'}{x} < 0 \quad \text{(inverted image)}$$

$$\boxed{M = \frac{x'}{x}} \qquad (3.25)$$

- If object and image space have different refractive indices,

3.23 has the more general form:

$$\frac{n'}{x'} + \frac{n}{x} = \frac{1}{f} \qquad (3.26)$$

3.8 Ray Tracing – thin lenses

- $x \to \infty \Rightarrow x' = n' f$
- $x' \to \infty \Rightarrow x = nf$

f = focal distance in air

- Let's differentiate (3.26) for air, n'=n=1:

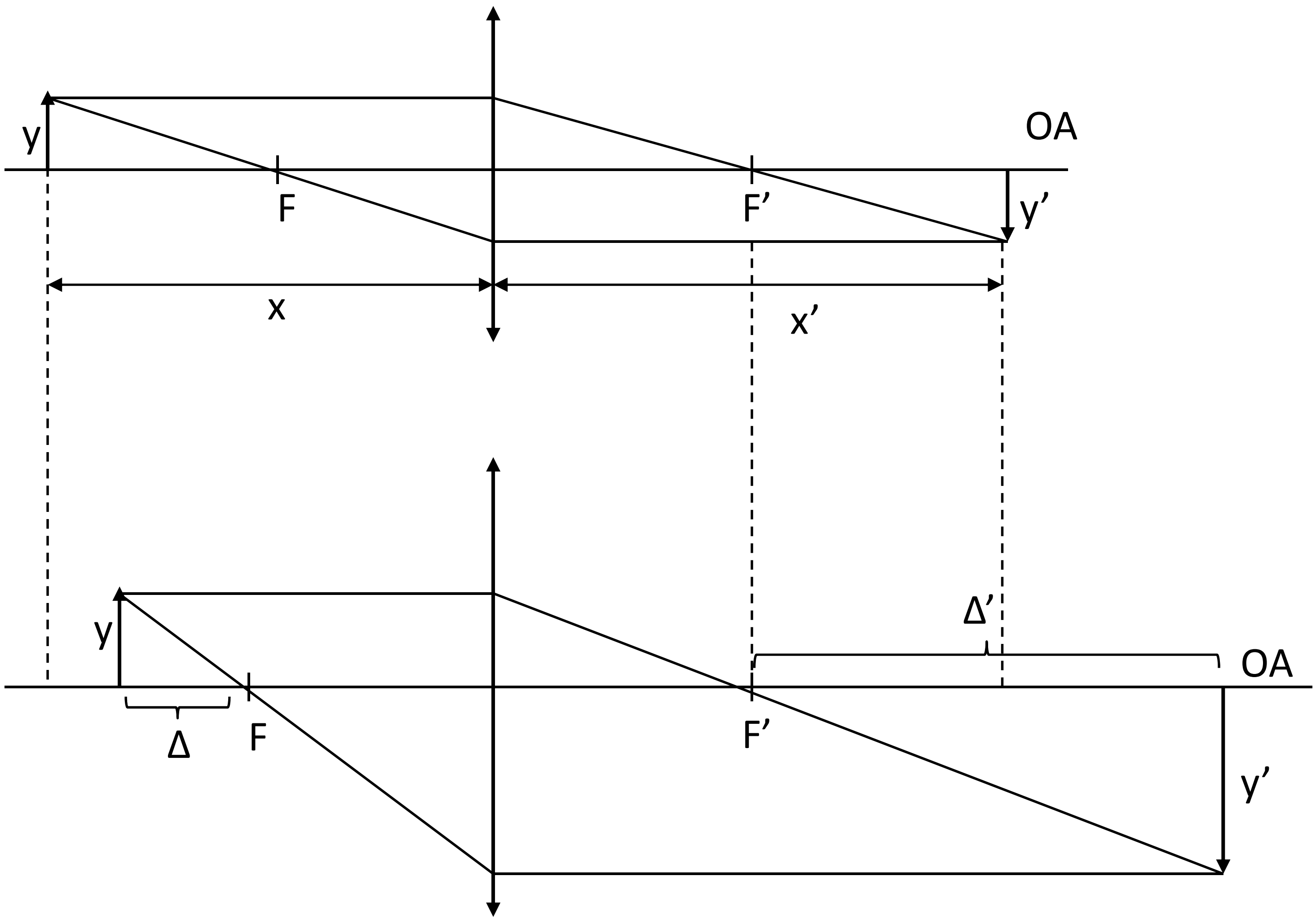
$$\frac{dx'}{x'^2} = -\frac{dx}{x^2}$$

$$dx' = -\left(\frac{x'}{x}\right)^2 dx$$

$$\boxed{dx' = -M^2 dx} \qquad (3.27)$$

- Eq 3.27 says that if the object gets closer to lens, the image moves away!

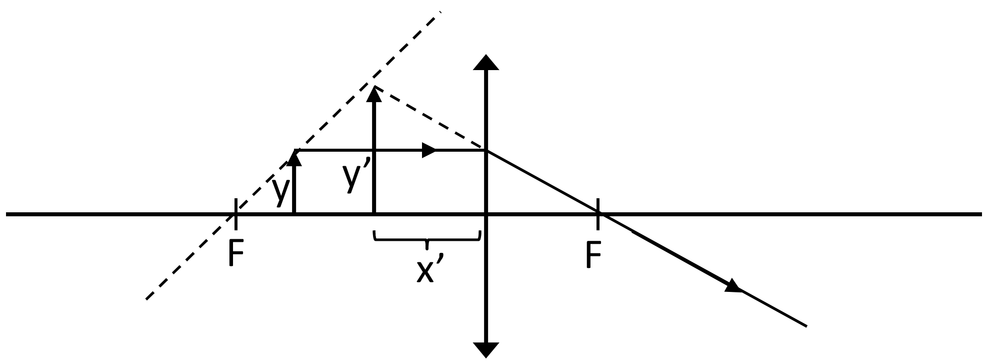
3.8 Ray Tracing – thin lenses



3.8 Ray Tracing – thin lenses

- What happens when x < f ?

$$\frac{1}{x'}+\frac{1}{x}=\frac{1}{f}\Rightarrow \boxed{\frac{1}{x'}=\frac{1}{f}-\frac{1}{x}<0} \ ?$$



- This image is formed by continuations of rays
    - Sometimes called "virtual images"
    - These images cannot be recorded directly (need re-imaging)

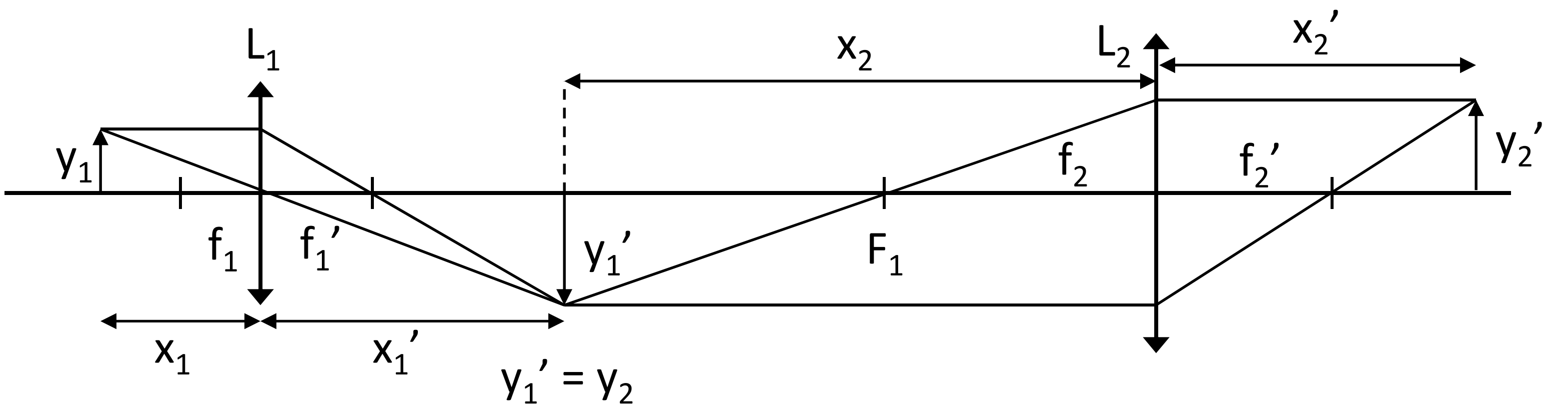
3.8 Ray Tracing – thin lenses

- Other useful formulas in G.O (figure above: Δ, Δ')
  - $\Delta\Delta' = f^2$ (Newton's formula)

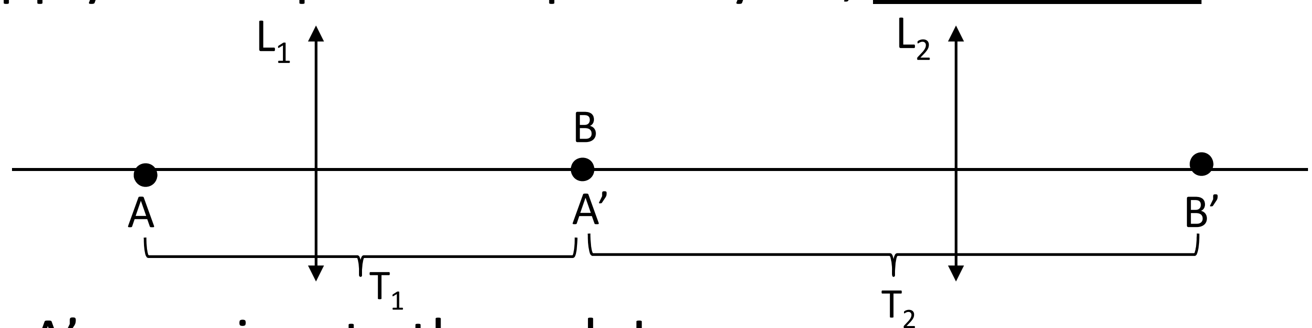
    (3.28)
  - $\frac{y'}{y} = \frac{\Delta'}{f} = \frac{f}{\Delta}$ ("lens formula")

3.9 System of lenses

- The image through one lens becomes object for the next lens, etc



- Apply lens equation repeteatly. Or, <u>use matrices</u>



- A, A' = conjugate through L_1
- B,B' = conjugate through L_2

3.9 System of lenses

- Use T = $T_2.T_1$; T matrix from 3.21

$$T_1 = \begin{pmatrix} 1-\frac{x_1'}{f_1} & 0 \\ -\frac{1}{f_1} & 1-\frac{x}{f} \end{pmatrix} \quad (3.29)$$

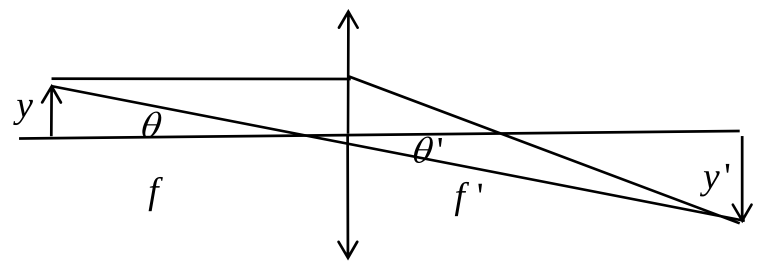
- <u>Note</u>: $1-\frac{x_1}{f_1} = 1-x_1\left(\frac{1}{x_1}+\frac{1}{x_1'}\right) =$

$$= 1-1-\frac{x_1}{x_1'} = \frac{1}{M_1} = \text{ magnification}$$

3.9 System of lenses

$$\rightarrow \begin{cases} 1-\dfrac{x_1}{f_1}=\dfrac{1}{M_1} \\ 1-\dfrac{x_1'}{f_1}=M_1 \end{cases} \quad (3.30)$$

also:



$\det(T_1) = 1$

$$\rightarrow T = T_2.T_1 =$$

Transverse Magnification

$$M = \frac{y'}{y}$$

$$= \begin{pmatrix} M_2 & 0 \\ -\dfrac{1}{f_2} & \dfrac{1}{M_2} \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ -\dfrac{1}{f_1} & \dfrac{1}{M_1} \end{pmatrix} =$$

$$\alpha = \frac{\theta'}{\theta} = \frac{y}{y'} = \frac{1}{M}$$

α

$$= \begin{pmatrix} M_1 M_2 & 0 \\ -\dfrac{M_1}{f_2} - \dfrac{1}{f_1 M_2} & \dfrac{1}{M_1 M_2} \end{pmatrix} \quad (3.31)$$

3.9 System of lenses

- 2-lens system is equivalent to:

$$\begin{cases} \dfrac{1}{f} = \dfrac{M_1}{f_2} + \dfrac{1}{f_1 M_2} \\ M = M_1 \cdot M_2 \end{cases}$$

- Microscopes achieve M=10-100 easily
- Can be reduced to 2-lens system
- Question: cascading many lenses such that $M=10^6$, would we be able to see atoms?
- Well, G.O can't answer that.
- So,back to wave optics