Homework 3 Solution

1.

a) Malus' Law : $I = I_0 \cos^2 \theta_i$ where θ_i is the angle between light polarization and axis of polarizer.

$\theta_i = \omega t + \theta_{initial}$

$I = I_0 \cos^2(\omega t + \theta_{initial})$

b) If two polarizers are present, the mutual angle between two axis gives the θ in Malus' law. As given above,

$I' = I \cos^2((\omega' - \omega)t + \theta'_{initial}) = I_0 \cos^2(\omega t + \theta_{initial}) \cos^2((\omega' - \omega)t + \theta'_{initial})$

Here we can see that the light is first filtered by the first polarizer and the second polarizer acts on the light already filtered by the first polarizer. Equation shows that the output of I will be similar to a) but will have an envelope with a frequency of $\omega' - \omega$.

c) From b) $(\omega' - \omega)t + \theta_{initial} = 90°$

Thus, I=0.

d) As the first polarizer is parallel to the incident light, we only have to think of polarizer 2 and 3.

From the angle between 2&3: $(\omega' - \omega)t + \theta_{initial} = 45°$

Angle between 1&2: $\omega t + \theta_{initial} = 45°$

$I' = I_0 \cos^2(45°) \cdot \cos^2(45°) = \frac{1}{4} I_0$

e) For all the adjacent polarizers axis differences are $\frac{90°}{N}$.

$I' = I_0 \cos^{2N}(\frac{90°}{N})$

If N is large $\lim_{N \to \infty} \cos^{2N}(\frac{90°}{N}) = 1$ (use Taylor expansion, $\cos^x(\frac{1}{x}) = 1 - \frac{1}{2x} + \frac{1}{8x^2} + \cdots$)

Thus, $I' = I_0$

2. Refer to www.wolframalpha.com or http://en.wikipedia.org/wiki/Lists_of_integrals for the calculation needed here.

$$E(t) = \exp[-\frac{t^2}{2\tau^2}]\exp[-i\omega_0 t]$$

$$\updownarrow$$

$$E(\omega) = \tau \exp[-\frac{\tau^2\omega^2}{2}] * \sqrt{2\pi}\delta(\omega + \omega_0) = \sqrt{2\pi}\tau \exp[-\frac{\tau^2(\omega+\omega_0)^2}{2}]$$

$$h(\omega) = \exp[i\beta\omega^2]$$

a)

$$F^{-1}[E(\omega)h(\omega)] = \frac{\sqrt{2\pi}\tau}{\sqrt{\tau^2 - 2i\beta}}\exp[-\frac{it(-4\omega_0\beta - 2i\omega_0\tau^2 + t)}{4\beta + 2i\tau^2}]$$

$$= \frac{\sqrt{2\pi}\tau^2}{\sqrt{\tau^2 - 2i\beta}}\exp[-\frac{t^2}{2(\tau^2 - 2i\beta)} + i\omega_0 t] = \frac{\sqrt{2\pi}\tau^2}{\sqrt{\tau^2 - 2i\beta}}\exp[-\frac{t^2}{2(\tau^2 + \frac{4\beta}{\tau^2})} + \frac{-i(t^2\beta + \omega_0 t(\tau^4 + 4\beta))}{(\tau^4 + 4\beta)}]$$

b)
before glass,

$$I(t) = \exp[-\frac{2t^2}{2\tau^2}] = \frac{1}{2}, t = \tau\sqrt{\ln 2}$$

After glass,

$$I(t) = \exp[-\frac{2t^2}{2(\tau^2 + \frac{4\beta}{\tau^2})}] = \frac{1}{2}$$

$$t^2 = \ln 2(\tau^2 + \frac{4\beta}{\tau^4}), t = \tau\sqrt{\ln 2(1 + \frac{4\beta}{\tau^2})}$$

Pulse broadens by a factor of $\sqrt{1 + \frac{4\beta}{\tau^2}}$

c)
before glass

$$E(\omega)E^*(\omega) = (\sqrt{2\pi}\tau \exp[-\frac{\tau^2(\omega+\omega_0)^2}{2}])^2 = 2\pi\tau^2 \exp[-\tau^2(\omega+\omega_0)^2]$$

$$F^{-1}[E(\omega)E^*(\omega)] = F^{-1}[\tau\sqrt{\pi}(\sqrt{2}\tau\exp[-\tau^2\omega^2]) * \sqrt{2\pi}\delta(\omega+\omega_0)]$$
$$= \frac{1}{\sqrt{2\pi}}\tau\sqrt{\pi}\exp[-\frac{t}{4\tau^2}]\exp[-i\omega_0 t] = \frac{\tau}{\sqrt{2}}\exp[-\frac{t}{4\tau^2}]\exp[-i\omega_0 t]$$

After glass,

$E(\omega)h(\omega)(E(\omega)h(\omega))^* = E(\omega)E^*(\omega)$

As $h(\omega)$ is purely imaginary.

Thus same result.

d)

Power spectrum $S(\omega) = |E(\omega)| = E(\omega)E^*(\omega)$.

identical problem.

f) $E(\omega)(E(\omega)h(\omega))^* = 2\pi\tau^2\exp[-\tau^2(\omega+\omega_0)^2]\exp[-i\beta\omega^2]$

$$F^{-1}[E(\omega)(E(\omega)h(\omega))^*] = \frac{\sqrt{2}\pi\tau^2}{\sqrt{\tau^2+i\beta}}\exp[-\frac{4i(\beta\omega_0{}^2\tau^2+\omega_0 t\tau^2)+t^2}{4(\tau^2+i\beta)}]$$

3. Here note that x,y are separate variables so you can calculate them separately.

$$E(u,v)=\left[\frac{ia^2b^2}{a^2+ib^2}\sqrt{\frac{1}{a^2}-\frac{i}{b^2}}\right]^2\exp[-\frac{ia^2b^2(u^2+v^2)}{2(a^2+ib^2)}]=\frac{a^2b^2}{b^2-ia^2}\exp[-\frac{ia^2b^2(u^2+v^2)}{2(a^2+ib^2)}]$$

a)

$$F^{-1}[E(u,v)h(u,v)]=\frac{1}{1-\frac{2\beta}{b^2}-\frac{2i\beta}{a^2}}\exp[\frac{x^2+y^2}{4(i\beta-\frac{ia^2b^2}{2(a^2+ib^2)})}]$$

$$=\frac{1}{1-\frac{2\beta}{b^2}-\frac{2i\beta}{a^2}}\exp[\frac{(x^2+y^2)}{4(i\beta-\frac{a^2b^4+ia^4b^2}{2(a^4+b^4)})}]$$

$$=\frac{1}{1-\frac{2\beta}{b^2}-\frac{2i\beta}{a^2}}\exp[\frac{(x^2+y^2)}{4(-\frac{a^2b^4}{2(a^4+b^4)}+i(\beta-\frac{a^4b^2}{2(a^4+b^4)}))}]$$

$$=\frac{1}{1-\frac{2\beta}{b^2}-\frac{2i\beta}{a^2}}\exp[\frac{(x^2+y^2)}{4\frac{a^2b^4}{2(a^4+b^4)}(-1+i(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2}))}]$$

$$=\frac{1}{1-\frac{2\beta}{b^2}-\frac{2i\beta}{a^2}}\exp[\frac{(x^2+y^2)(-1-i(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2}))}{4\frac{a^2b^4}{2(a^4+b^4)}(1+(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2})^2)}]$$

b) Symmetry applies for x and y direction so calculate for x.
Before microscope,

$$\frac{I(x)}{I(0)}=\exp[-\frac{x^2}{a^2}]=\frac{1}{2},\ x=\sqrt{\ln 2}a$$

After microscope,

$$\frac{I(x)}{I(0)}=\exp[-\frac{x^2}{\frac{a^2b^4}{(a^4+b^4)}(1+(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2})^2)}]=\frac{1}{2},$$

$$x=\sqrt{\ln 2\frac{a^2b^4}{(a^4+b^4)}(1+(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2})^2)}=a\sqrt{\ln 2}\sqrt{\frac{b^4}{(a^4+b^4)}(1+(\frac{2(a^4+b^4)}{a^2b^4}\beta-\frac{a^2}{b^2})^2)}$$

c) $E(u,v)E^*(u,v) = \frac{a^4b^4}{b^4+a^4}\exp[-\frac{a^2b^4(u^2+v^2)}{(a^4+b^4)}]$, let $\frac{1}{4k} = \frac{a^2b^4}{a^4+b^4}$

$$E(u,v)E^*(u,v) = \frac{a^2}{2}\frac{1}{2k}\exp[-\frac{(u^2+v^2)}{4k}]$$

$$F^{-1}[E(u,v)E^*(u,v)] = \frac{a^2}{2}\exp[-\frac{a^4+b^4}{4a^2b^4}(x^2+y^2)]$$

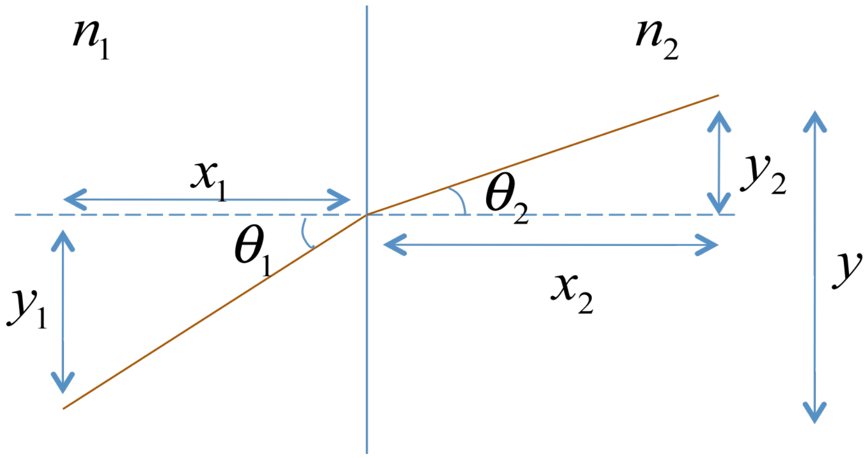
d) Same as c as in #2

e) $E(u,v)E^*(u,v)h^*(u,v) = \frac{a^4b^4}{b^4+a^4}\exp[(-\frac{a^2b^4}{(a^4+b^4)}+i\beta)(u^2+v^2)]$

$$F^{-1}[E(u,v)E^*(u,v)h^*(u,v)] = \frac{1}{\frac{2a^2b^4}{a^4+b^4}-2i\beta}\exp[\frac{x^2+y^2}{4(-\frac{a^2b^4}{a^4+b^4}+i\beta)}]$$

4. #2 and #3 shows broadening in temporal domain and spatial domain. This is why calculation for both problems are almost identical.

5. Fermat's principle : minimize optical path length



Optical path length : $l = \sqrt{x_1^2 + y_1^2} \bullet n_1 + \sqrt{x_2^2 + y_2^2} \bullet n_2 = \sqrt{x_1^2 + y_1^2} \bullet n_1 + \sqrt{x_2^2 + (y - y_1)^2} \bullet n_2$

$$\frac{dl}{dy_1} = \frac{2y_1 n_1}{2\sqrt{x_1^2 + y_1^2}} + \frac{-2(y - y_1)n_2}{2\sqrt{x_2^2 + (y - y_1)^2}} = 0$$

$$\frac{y_1 n_1}{l_1} = \frac{y_2 n_2}{l_2}, n_1 \sin\theta_1 = n_2 \sin\theta_2$$