

THE WAVE EQUATION

5.1. Solution to the wave equation in Cartesian coordinates

Recall the Helmholtz equation for a scalar field U in rectangular coordinates

$$\nabla^2 U(\mathbf{r}, \omega) + \beta^2(\mathbf{r}, \omega)U(\mathbf{r}, \omega) = 0, \quad (5.1)$$

Where β is the wavenumber, defined as

$$\begin{aligned} \beta^2(\mathbf{r}, \omega) &= \omega^2 \mu \varepsilon(\mathbf{r}, \omega) - i\omega \mu \sigma(\mathbf{r}, \omega) \\ &= n^2(\mathbf{r}, \omega) \frac{\omega^2}{c^2} - i\omega \mu \sigma(\mathbf{r}, \omega) \end{aligned} \quad (5.2)$$

Assuming lossless medium ($\sigma = 0$) and decoupling the vacuum contribution ($n = 1$) from $\beta(\mathbf{r}, \omega)$, we re-write Eq. 2 to explicitly show the driving term, namely

$$\nabla^2 U(\mathbf{r}, \omega) + k_0^2 U(\mathbf{r}, \omega) = -k_0^2 [n^2(\mathbf{r}, \omega) - 1] U(\mathbf{r}, \omega), \quad (5.3)$$

where $k_0 = \omega/c$. Note that Eq. 3 preserves the generality of the Helmholtz equation. The Green's function, h , (an impulse response) is obtained by setting the driving term (the right hand side) to a delta function,

$$\begin{aligned} \nabla^2 h(\mathbf{r}, \omega) + k_0^2 h(\mathbf{r}, \omega) &= -\delta^{(3)}(\mathbf{r}) \\ &= -\delta(x)\delta(y)\delta(z) \end{aligned} \quad (5.4)$$

In order to solve this equation, we take the Fourier transform with respect to \mathbf{r} ,

$$-k^2 \tilde{h}(\mathbf{k}, \omega) + k_0^2 \tilde{h}(\mathbf{k}, \omega) = -1, \quad (5.5)$$

where $k^2 = \mathbf{k} \cdot \mathbf{k}$.

This equation breaks into three identical equations, for each spatial coordinate, as

$$\begin{aligned}\tilde{h}(k_x, \omega) &= \frac{1}{k_x^2 - k_0^2} \\ &= \frac{1}{2k_{0x}} \left[\frac{1}{k_x - k_0} - \frac{1}{k_x + k_0} \right]\end{aligned}\tag{5.6}$$

To calculate the Fourier transform of Eq. 6, we invoke the shift theorem and the Fourier transform of function $1/k$,

$$\begin{aligned}f(k-a) &\rightarrow e^{iax} \cdot \tilde{f}(x) \\ \frac{1}{k} &\rightarrow i \operatorname{sign}(x)\end{aligned}\tag{5.7}$$

Thus we obtain as the final solution

$$h(x, \omega) = \frac{1}{ik_0} \cdot e^{ik_0x}, \quad x > 0\tag{5.8}$$

The procedure applies to all three dimensions, such that the 3D solution reads

$$h(\mathbf{r}, \omega) \propto e^{ik_0 \hat{\mathbf{k}} \cdot \mathbf{r}},\tag{5.9}$$

where $\hat{\mathbf{k}}$ is the unit vector, $\hat{\mathbf{k}} = \mathbf{k} / k$. Equation 9 describes the well known *plane wave* solution, which is characterized by the absence of amplitude modulation upon propagation. This is an infinitely broad wavefront that propagate along direction $\hat{\mathbf{k}}$ (Figure 1).

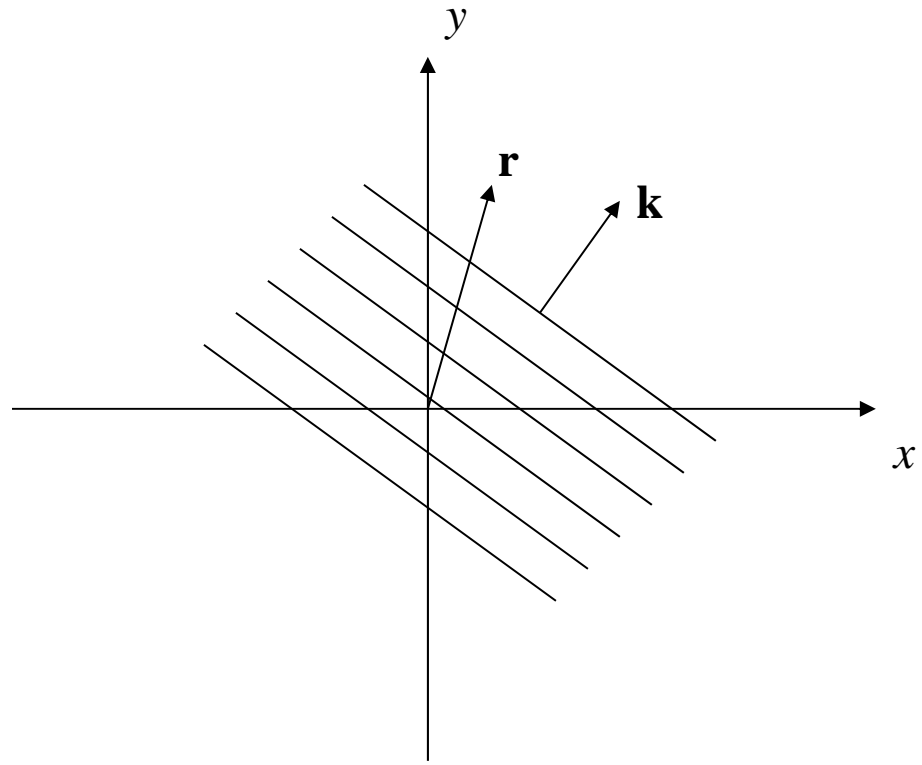


Figure 5-1. Plane wave.

5.2. Solution of the wave equation in spherical coordinates

For light propagation with spherical symmetry, such as emission from a point source in free space, the problem becomes one dimensional, with the radial coordinate as the only variable,

$$\begin{aligned}
 h(\mathbf{r}, \omega) &= U(r, \omega) \\
 \tilde{h}(\mathbf{k}, \omega) &= \tilde{U}(k, \omega) \\
 {}^{(3)}\delta(\mathbf{r}) &= \frac{1}{2\pi r^2} {}^{(1)}\delta(r) \\
 {}^{(3)}\delta(\mathbf{r}) &\rightarrow 1
 \end{aligned} \tag{5.10}$$

Recall that the Fourier transform pair is defined in this case as

$$\begin{aligned}
 h(r) &\propto \int_0^\infty \tilde{h}(k) \cdot \text{sinc}(kr) k^2 dk \\
 \tilde{h}(k) &\propto \int_0^\infty h(r) \cdot \text{sinc}(kr) r^2 dr
 \end{aligned} \tag{5.11}$$

The Fourier properties of ${}^{(3)}\delta(r)$ and ∇^2 extend naturally to the spherically symmetric case as

$$\begin{aligned}
 {}^{(3)}\delta(r) &\rightarrow 1 \\
 \nabla^2 &\rightarrow -k^2
 \end{aligned} \tag{5.12}$$

Thus, by Fourier transforming Eq. 4, we obtain the frequency domain solution,

$$\tilde{h}(k, \omega) = \frac{1}{k^2 - k_0^2} \tag{5.13}$$

Not surprisingly, the frequency domain solutions for the Cartesian and spherical coordinates (Eqs. 6 and 13, respectively) look quite similar, except that the former depends on one component of the wave vector and the latter on the modulus of the wave vector. The solution in the spatial domain becomes

$$\begin{aligned}
 h(r, \omega) &= \int_0^{\infty} \frac{1}{k^2 - k_0^2} \cdot \frac{\sin(kr)}{kr} \cdot k^2 dk \\
 &= \frac{1}{2r} \int_0^{\infty} \left(\frac{1}{k - k_0} + \frac{1}{k + k_0} \right) \frac{e^{ikr} - e^{-ikr}}{2i} dk \\
 &= \frac{1}{4ir} \int_0^{\infty} \frac{e^{ikr}}{k - k_0} dk.
 \end{aligned} \tag{5.14}$$

We recognize in Eq. 15 the Fourier transform of a shifted $1/k$ function, which we encountered earlier (Eqs. 7). Thus, evaluating this Fourier transform, we finally obtain Green's function for propagation from a point source,

$$h(r, \omega) \propto \frac{e^{ikr}}{r}, \quad r > 0 \tag{5.15}$$

This solution defines a (outgoing) spherical wave.

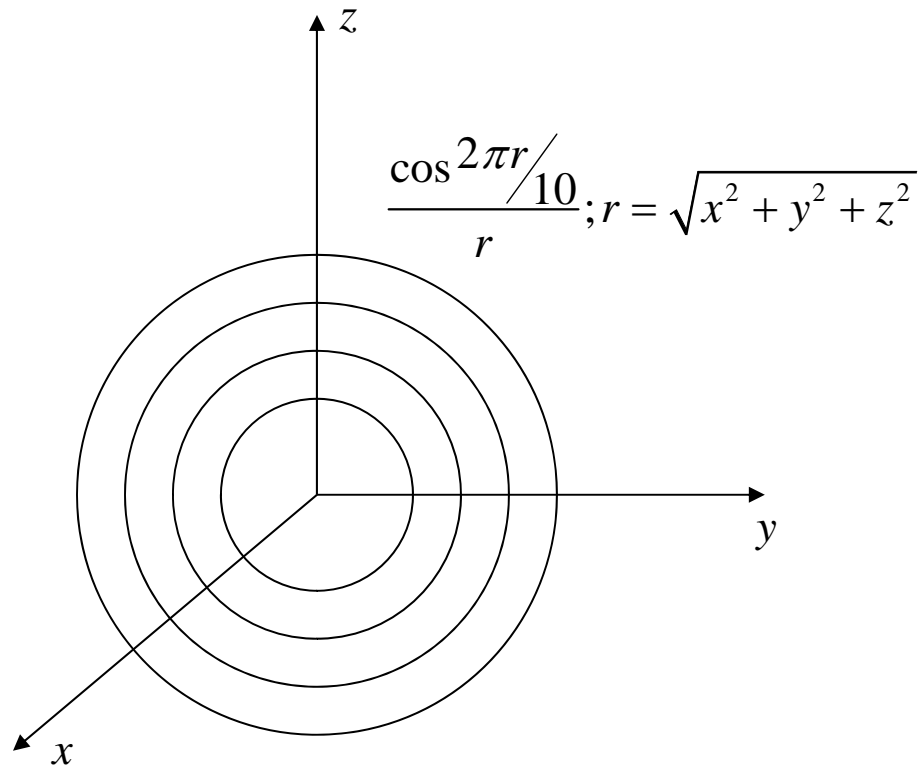
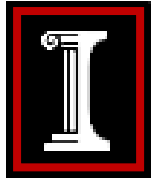
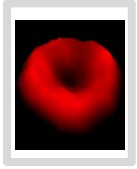


Figure 5-2. Spherical wave



Chapter 6 – Fourier Optics

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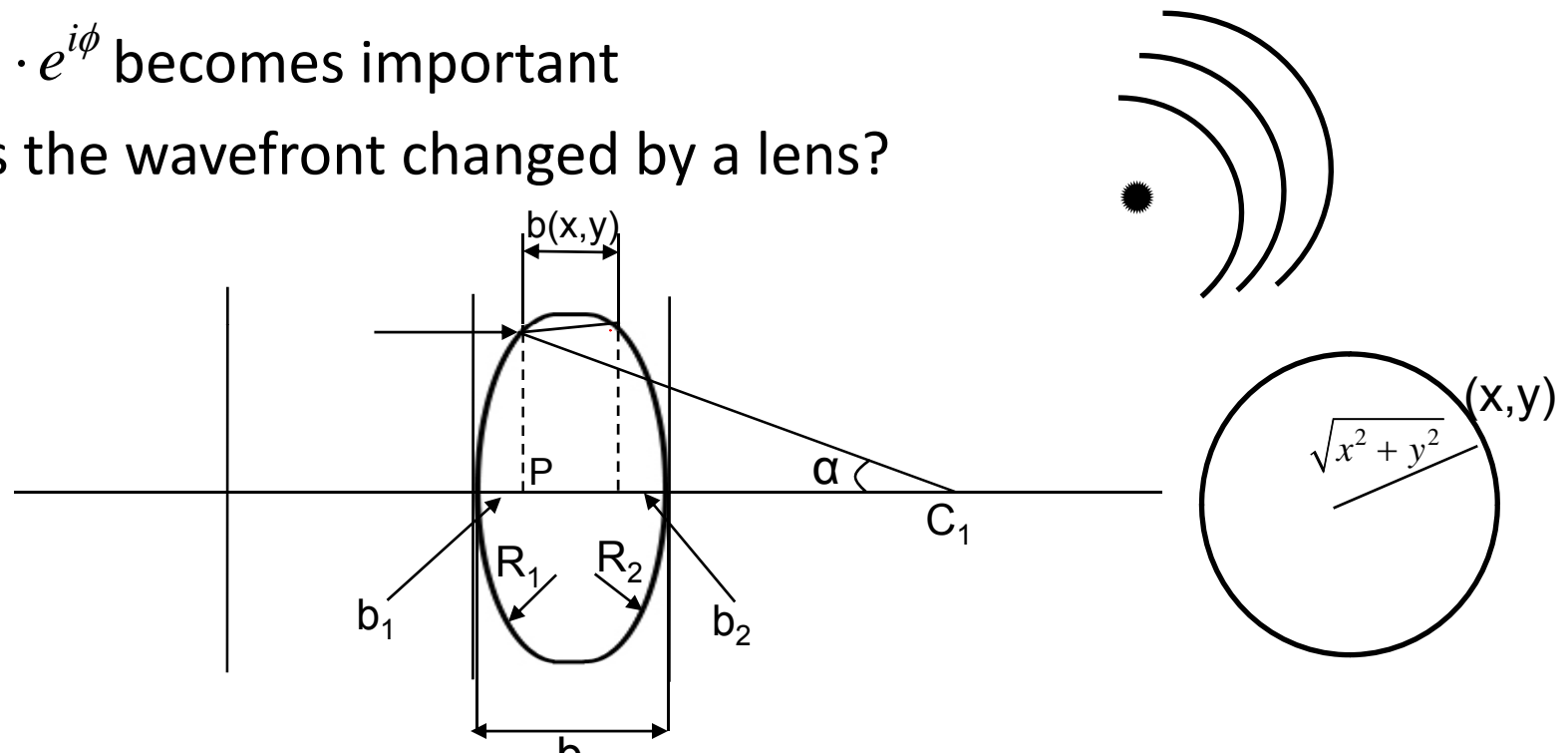
Quantitative Light Imaging Laboratory

<http://light.ece.uiuc.edu>



3.10 Lens as a phase transformer

- $E = E_o \cdot e^{i\phi}$ becomes important
- How is the wavefront changed by a lens?



$$\phi(x, y) = \underbrace{knb(x, y)}_{\text{glass}} + \underbrace{k[b_o - b(x, y)]}_{\text{air}} \quad (3.32)$$

$$= kb_o + k(n-1)b(x, y)$$



3.10 Lens as a phase transformer

- Let's calculate $b(x,y)$; assume small angles

$$\begin{aligned}
 b_1 &= R_1 - (PC_1) = \\
 &= R_1 - \sqrt{R_1^2 - (\alpha R_1)^2} = R_1 \left[1 - \sqrt{1 - \alpha^2} \right]
 \end{aligned}$$

- Taylor expansion: $\sqrt{1+x} \big|_{x \rightarrow 0} \approx 1 + \frac{x}{2}$

Small Angle Approx
(Gaussian)

$$\rightarrow b_1 = R_1 \left[1 - \left(1 - \frac{\alpha^2}{2} \right) \right] = R_1 \frac{\alpha^2}{2} \tag{3.33}$$

- $\alpha \approx \tan \alpha = \frac{\sqrt{x^2 + y^2}}{R_1}$

- So: $b_1(x, y) = \frac{x^2 + y^2}{2R_1}$ (3.34)



3.10 Lens as a phase transformer

$$\begin{aligned} \rightarrow b(x, y) &= b_o - b_1(x, y) - b_2(x, y) = \\ &= b_o - \frac{x^2 + y^2}{2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \end{aligned} \quad (3.35)$$

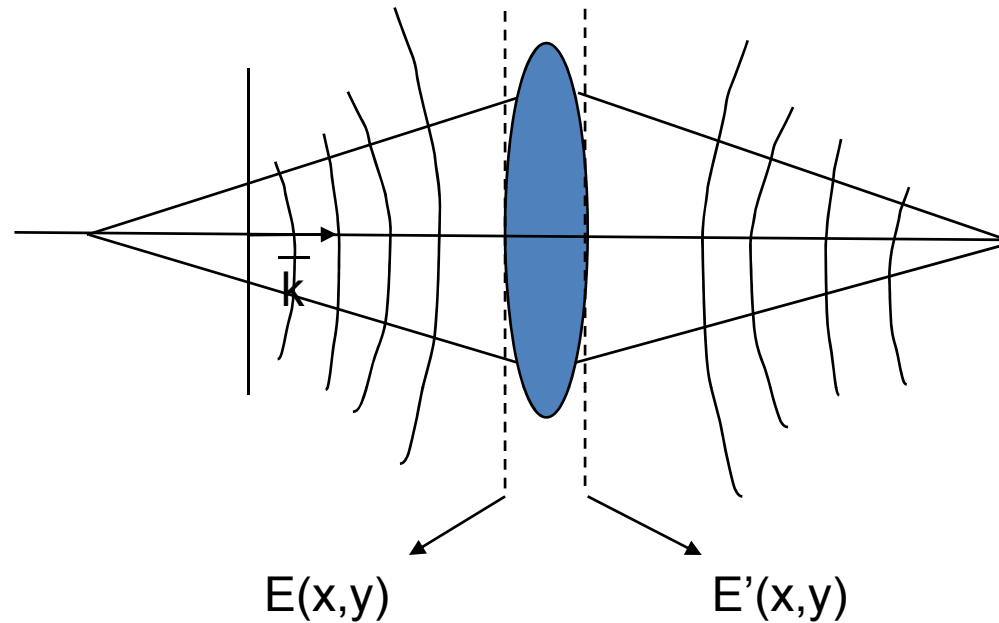
- This is the thickness approximation
- The phase ϕ becomes:

$$\begin{aligned} \phi(x, y) &= \phi_o - k(n-1)b(x, y) = \\ &= \phi_o - k \frac{x^2 + y^2}{2} (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right] \end{aligned} \quad (3.36)$$

- But we know: $\frac{1}{f} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$



3.10 Lens as a phase transformer



$$\rightarrow E'(x, y) = E(x, y) \cdot t_e(x, y) \quad (3.37)$$

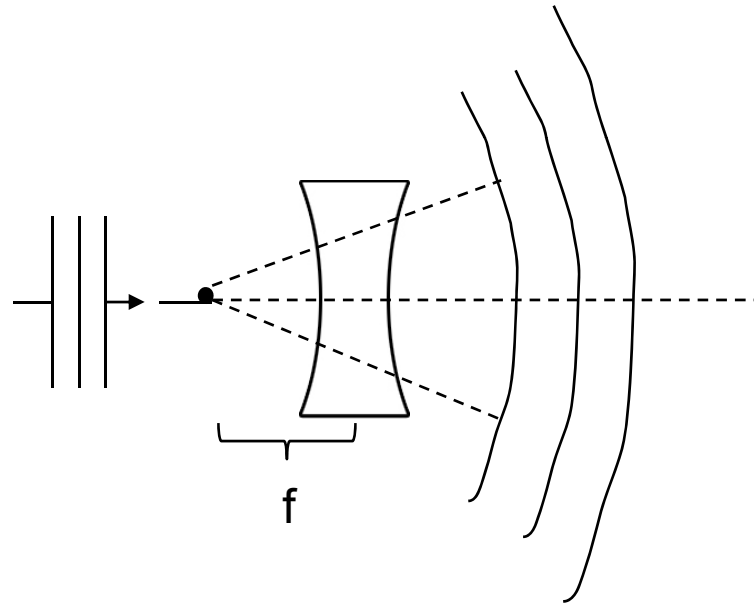
- The lens transformation is:

$$\begin{aligned} t_e &= e^{i\phi} = \\ &= e^{iknb_0} \cdot e^{-i\frac{k}{2f}(x^2+y^2)} \end{aligned} \quad (3.38)$$



3.10 Lens as a phase transformer

- A lens transforms an incident plane wavefront into a parabolic shape
- Note: $f > 0$ convergent lens
 $f < 0$ divergent



- So, if we know how to propagate through free space, then we can calculate field amplitude and phase through any imaging system

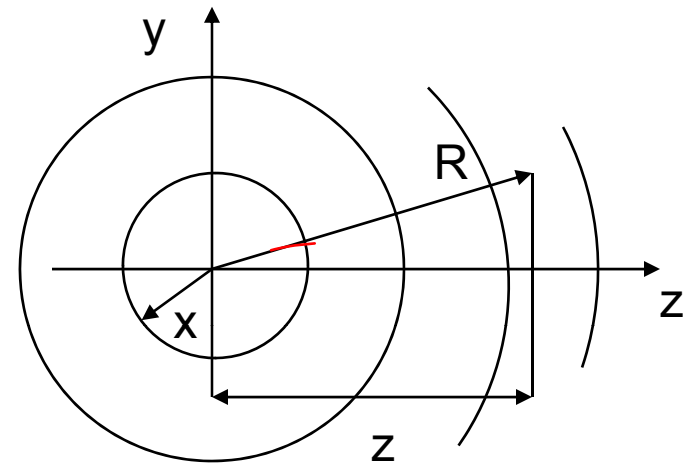


3.11 Huygens-Fresnel principle

- Spherical waves:

- Wavelet:
$$h = \frac{e^{ikR}}{R}$$

- $$R = \sqrt{x^2 + y^2 + z^2} = z \sqrt{1 + \frac{x^2 + y^2}{z^2}}$$



- We are interested close to OA, i.e. small angles

$$\rightarrow R \simeq z \left[1 + \frac{1}{2} \left(\frac{x^2 + y^2}{z^2} \right) \right] \quad (3.39)$$



3.11 Huygens-Fresnel principle

- For amplitude $\frac{1}{R} \approx \frac{1}{z}$ is OK

$$e^{ikR}$$

$$R \approx z$$

- For phase $kR \approx kz \left[1 + \frac{1}{z} \left(\frac{x^2 + y^2}{z} \right) \right]$



→ The wavelet becomes:

$$h(x, y) \approx \frac{e^{ikz}}{z} e^{i \frac{k(x^2 + y^2)}{2z}} \quad (3.40a)$$

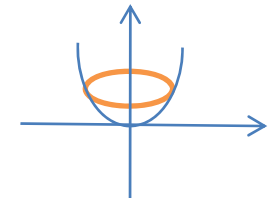
$$f(x, y) = x^2 + y^2$$

- Remember, for the lens we found:

$$t_e(x, y) = e^{i\phi_0} e^{i \frac{k}{2f} (x^2 + y^2)} \quad (3.40b)$$

← Negative Lens

$$f(x) = x^2$$

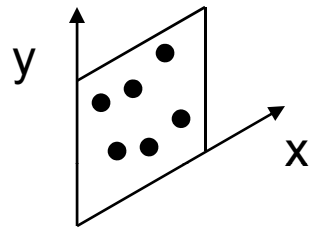


- Free space acts on the wavefront like a divergent lens
(note “+” sign in phase)



3.11 Huygens-Fresnel principle

- At a given plane, a field is made of point sources



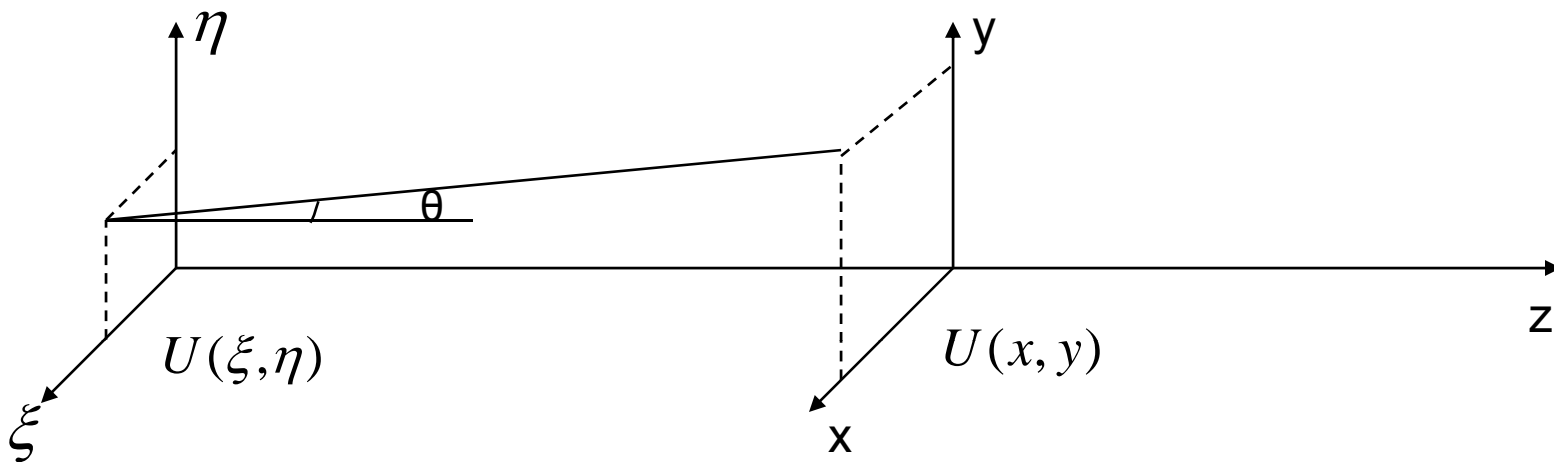
$$E(x, y) = \iint E(x', y') \delta(x - x') \delta(y - y') dx' dy'$$

- Eq 3.40 a-b represent the impulse response of the system (free space or lens)
- Recall linear systems (Chapter 2, page 12, Eq 2.16)
 - Final response (output) is the convolution of the input with the impulse response (or Green's function)
- Nice! Space or time signals work the same!

$$\int f(x') \delta(x_0 - x') dx' \Rightarrow f(x_0)$$

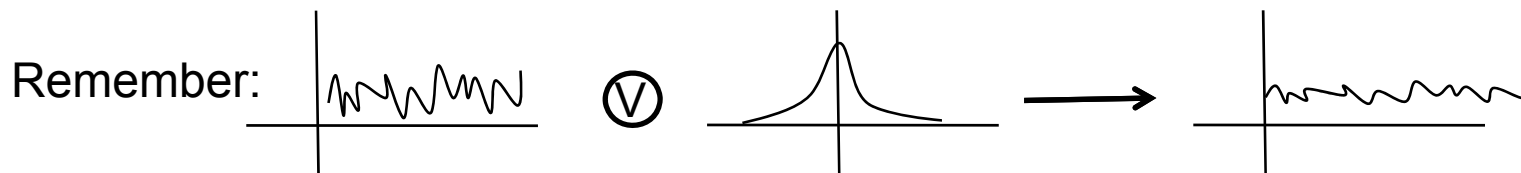


3.11 Huygens-Fresnel principle



$$U(x, y) = \iint U(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta$$

$$U(x, y) = \iint U(\xi, \eta) e^{\frac{ik}{2z} [(x-\xi)^2 + (y-\eta)^2]} d\xi d\eta \quad (3.41)$$





3.11 Huygens-Fresnel principle

- Fresnel diffraction equation = convolution
- Fresnel diffraction equation is an approximation of Huygens principle (17th century) $\left(R = z \left[1 + \frac{x^2 + y^2}{2z^2} \right] \right)$

$$U(x, y) = \frac{1}{i\lambda} \iint U(\xi, \eta) \frac{e^{ikR(\xi, \eta)}}{R(\xi, \eta)} \cos \theta(\xi, \eta) d\xi d\eta \quad (3.42)$$

- ! Fresnel is good enough for our purpose
- Note: we don't care about constants A (no x-y dependence)

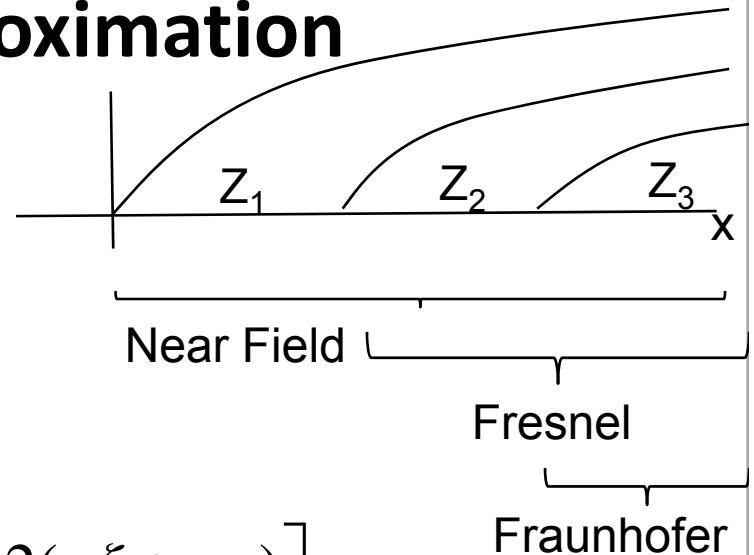
$$U(x, y) = \iint U(\xi, \eta) e^{\frac{ik}{2z} [(x-\xi)^2 + (y-\eta)^2]} d\xi d\eta$$



3.12 Fraunhofer Approximation

- One more approximation (far field)
- The phase factor in Fresnel is:

$$\begin{aligned}
 \phi(x, y) &= \frac{k}{2z} \left[(x - \xi)^2 + (y - \eta)^2 \right] = \\
 &= \frac{k}{2z} \left[\underbrace{(x^2 + y^2)}_{\approx 0} + (\xi^2 + \eta^2) - 2(x\xi + y\eta) \right] \quad (3.43)
 \end{aligned}$$



- If $z \gg k(\xi^2 + \eta^2)$, we obtain the Fraunhofer equation:

$$U(x, y) = A \iint_{-\infty}^{\infty} U(\xi, \eta) e^{-\frac{i2\pi}{\lambda z}(x\xi + y\eta)} d\xi d\eta \quad (3.44)$$

- Thus eq. 3.44 defines a Fourier transform
- Useful to calculate diffraction patterns !

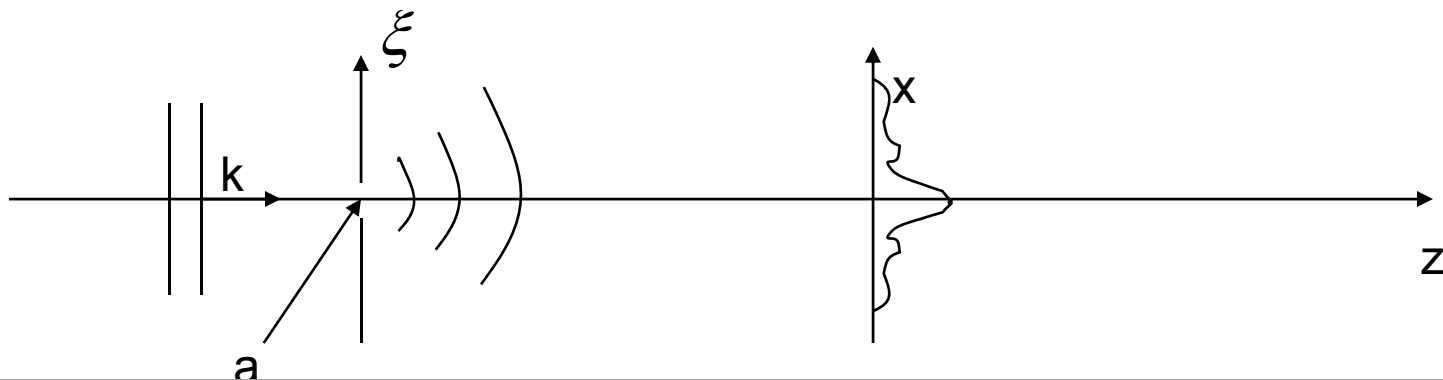


3.12 Fraunhofer Approximation

- Let's define:
$$\begin{cases} f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z} \end{cases}$$

$$\rightarrow U(f_x, f_y) = \iint_{-\infty}^{\infty} U(\xi, \eta) \cdot e^{-i2\pi(\xi f_x + \eta f_y)} d\xi d\eta \quad (3.45)$$

- Example: diffraction on a slit





3.12 Fraunhofer Approximation

- One dimensional: $U(x) = \Pi\left(\frac{x}{a}\right) = \begin{cases} a, & |x| < a/2 \\ 0, & \text{rest} \end{cases}$

- The far-field is given by Fraunhofer eq:

$$U(f_x) = \int_{-\infty}^{\infty} U(x) e^{-i2\pi x f_x} dx =$$

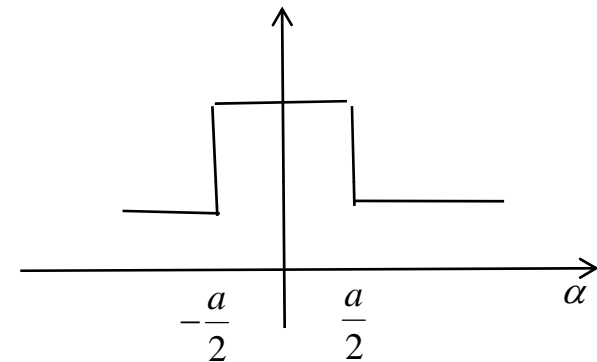
$$= \mathfrak{F}\left[\Pi\left(\frac{x}{a}\right)\right] =$$

- Similarity Theorem + $\mathfrak{F}[\Pi(x)] = \text{sinc}(f_x)$:

$$\rightarrow U(f_x) = a \text{sinc}(af_x) =$$

$$= a \frac{\text{sinc}(af_x)}{af_x} \quad (3.46)$$

- $f_x = \frac{x}{\lambda z}$





3.12 Fraunhofer Approximation

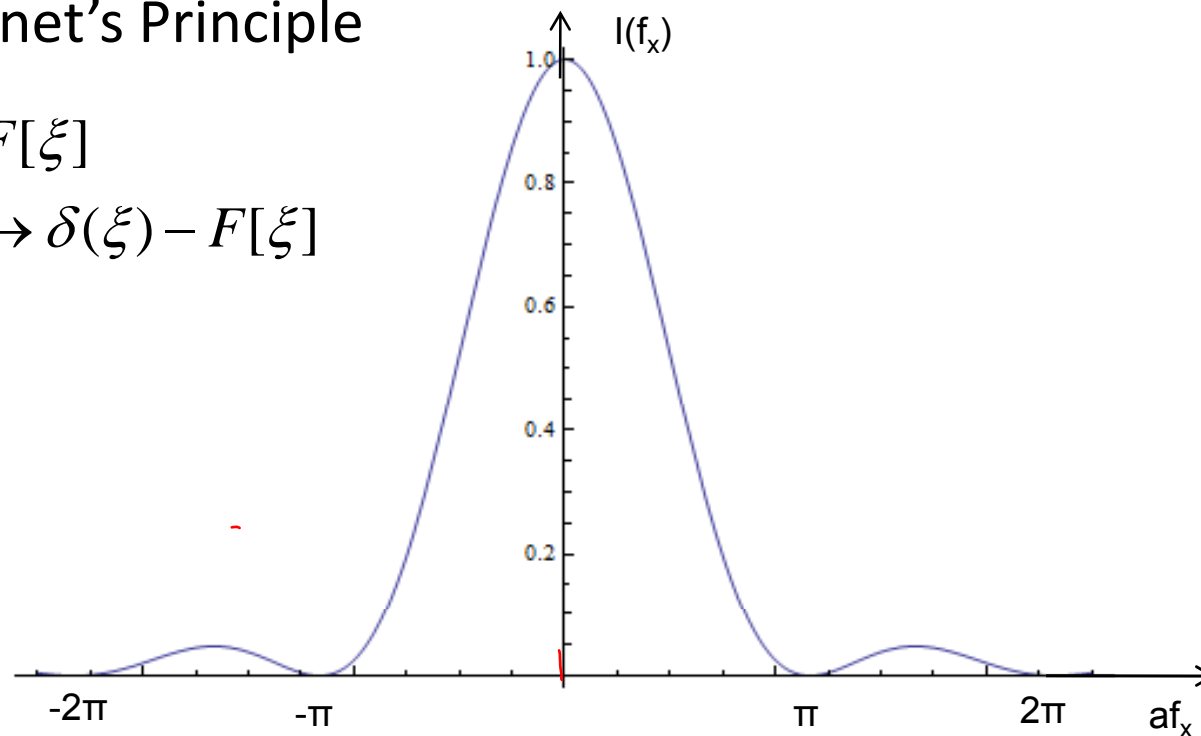
- Always measure intensity \rightarrow the diffraction pattern is:

$$I(f_x) = |U(f_x)|^2 = a^2 \left[\frac{\sin(af_x)}{af_x} \right]^2 \quad (3.48)$$

- Also Babinet's Principle

$$f(x) \rightarrow F[\xi]$$

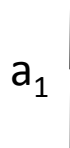

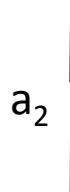
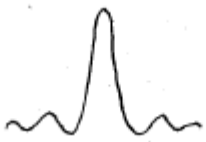
$$1 - f(x) \rightarrow \delta(\xi) - F[\xi]$$





3.12 Fraunhofer Approximation

- Note: $\sin(af_x) = \sin\left(\frac{f_x}{\frac{1}{a}}\right) \rightarrow \frac{1}{a} = \text{width of diffraction pattern}$

- narrow slit: a_1  \rightarrow 
- wide slit: a_2  \rightarrow 

- Similarity Theorem \leftrightarrow uncertainty principle

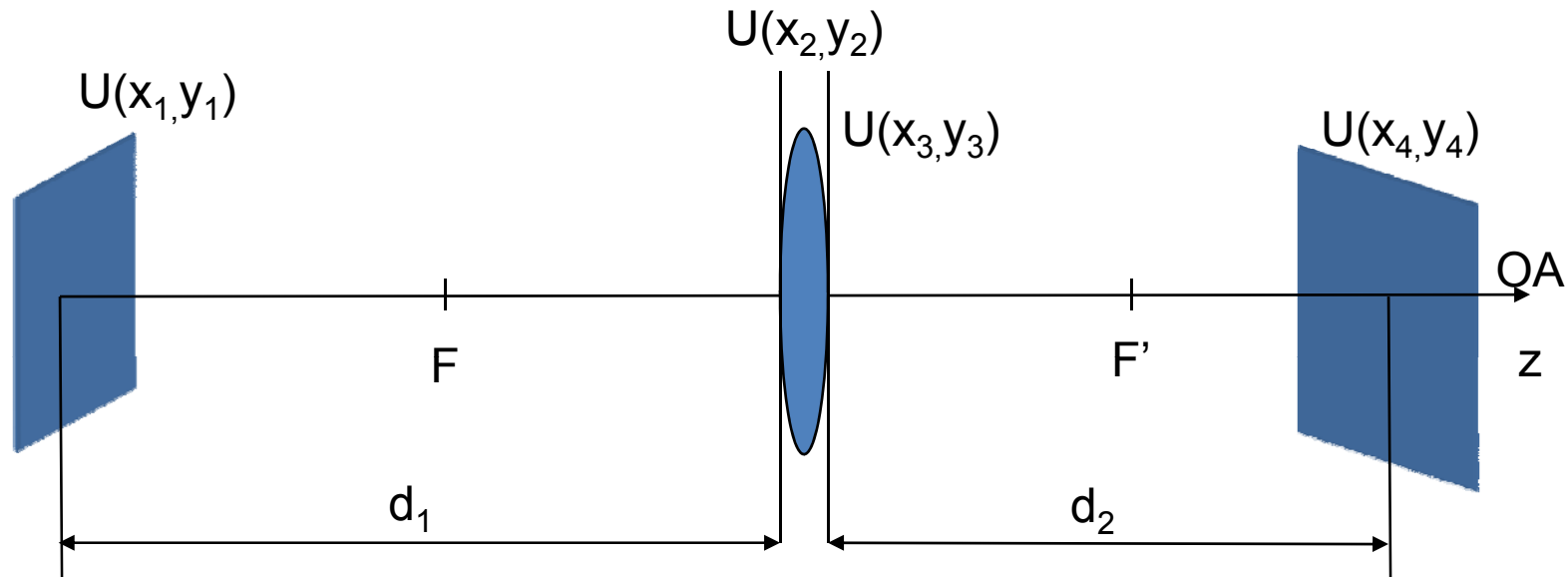
Quiz:

What is the diffraction pattern from 2 slits of size a separated by d ?

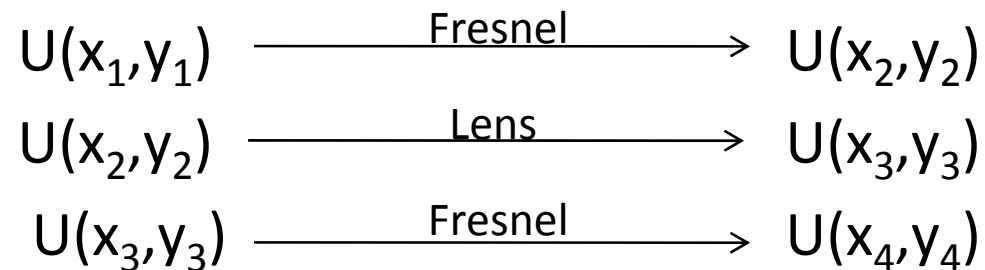




3.13 Fourier Properties of lenses



- Propagation:





3.13 Fourier Properties of lenses

- $$U(x_2, y_2) = A_{12} \iint U(x_1, y_1) e^{\frac{ik}{2d_1} [(x_2 - x_1)^2 + (y_2 - y_1)^2]} dx_1 dy_1$$
- $$U(x_3, y_3) = A_{23} U(x_3, y_3) e^{-i \frac{k}{2f} [x_3^2 + y_3^2]}$$
- $$U(x_4, y_4) = A_{34} \iint U(x_3, y_3) e^{\frac{ik}{2d_2} [(x_4 - x_3)^2 + (y_4 - y_3)^2]} dx_3 dy_3$$

(3.49)

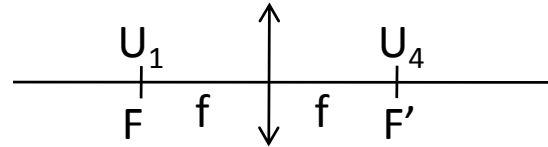
- Combining Eqs (3.49) is a little messy, but there is a special case when eqs simplify → very useful

|
.
?



3.13 Fourier Properties of lenses

- If $d_1 = d_2 = f$



$$\left\{ \begin{array}{l} U(x_4, y_4) = A_{41} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_1, y_1) e^{-i2\pi(x_1 f_x + y_1 f_y)} dx_1 dy_1 \\ f_x = \frac{x_4}{\lambda f}; f_y = \frac{y_4}{\lambda f} \end{array} \right. \quad (3.50)$$

- Same eq as (3.45); now $z \rightarrow f$
- **Lenses work as Fourier transformers**
 - Useful for spatial filtering

What is $|\mathfrak{F}[U]|^2$

$$\mathfrak{F}[|U|^2] =$$

$$\mathfrak{F}[U \cdot U^*] =$$

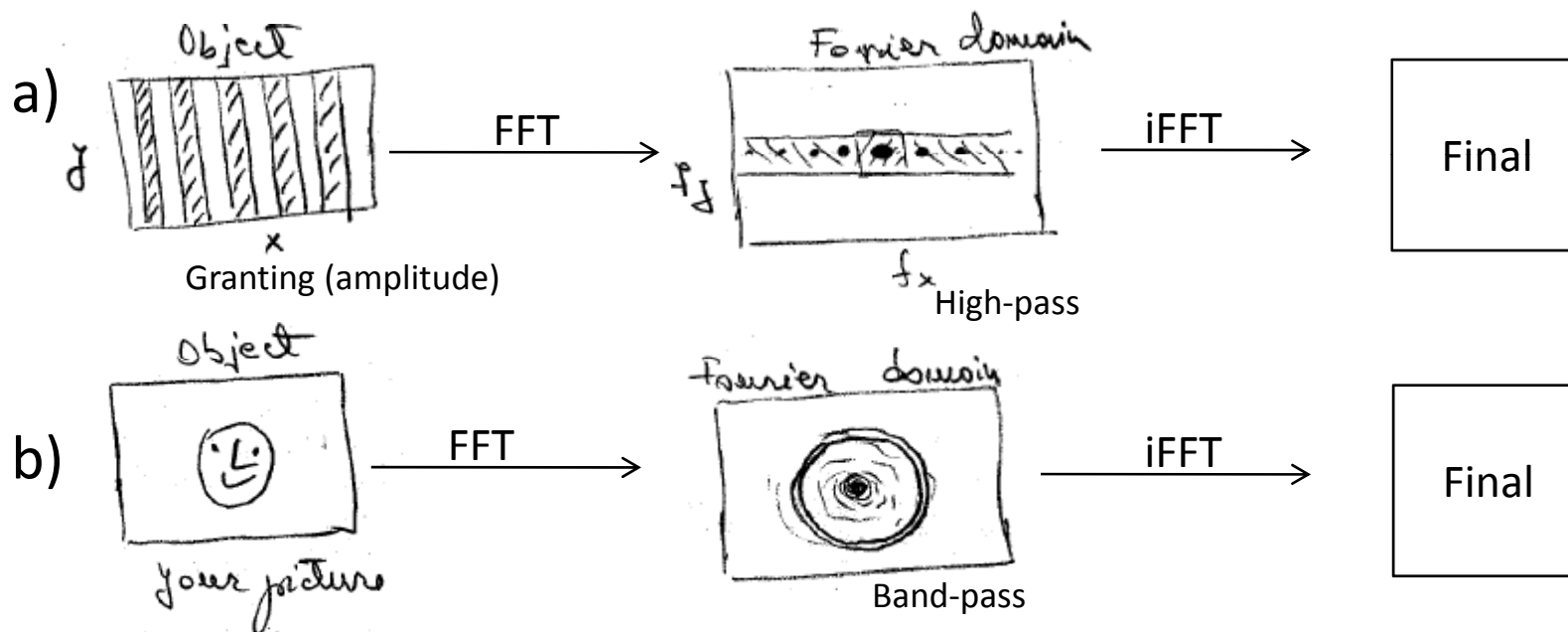
$$\tilde{U} \otimes \tilde{U}^*$$

Autocorrelation



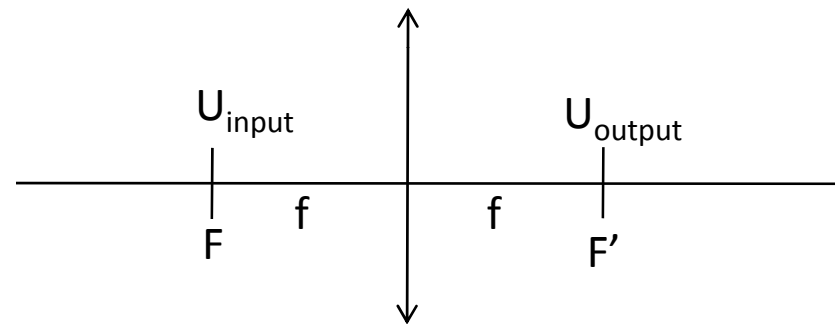
3.13 Fourier Properties of lenses

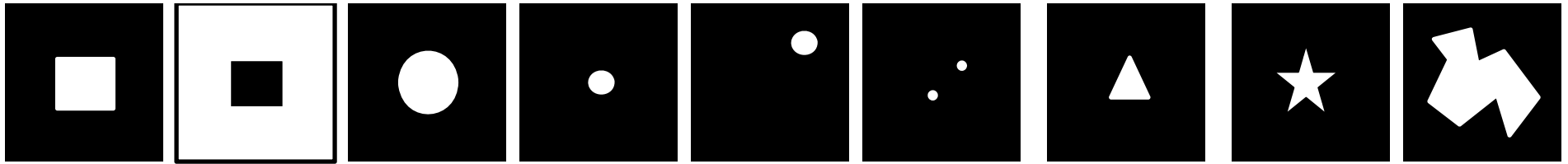
- Exercise: Use Matlab to FFT images (look up “fft2” in help)



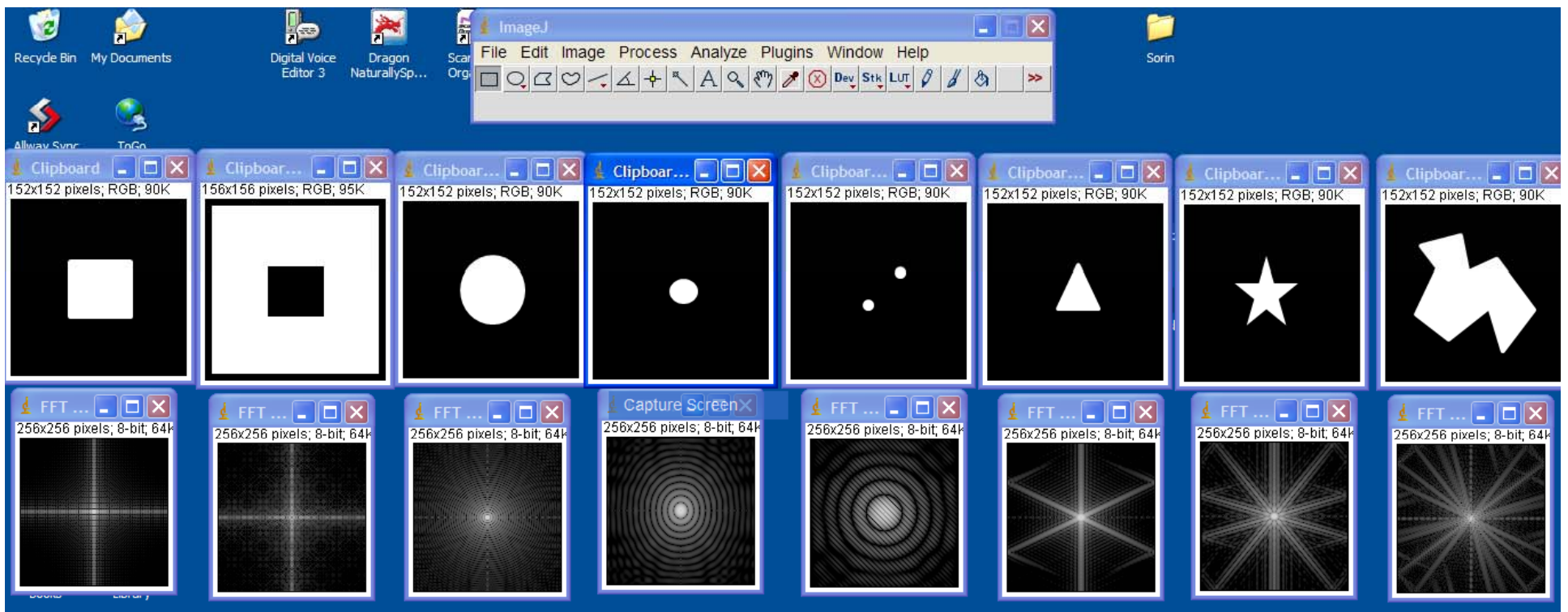
- Note the relationship between the frequencies passed and the details / contrast in the final image

Fourier Optics





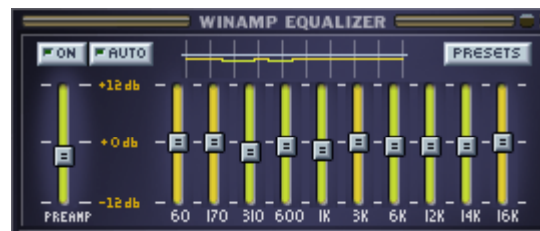
2D FT pairs \rightarrow diffraction patterns
(Using ImageJ)



FT is in $\text{Log}(\text{Abs})$ scale!! Why?

Time-domain: sound

- load your mp3
- plot time-series
- plot frequency amplitude, phase, power spectrum, linear/ log
- show frequency bands, i.e. “equalizer”
- adjust and play in real time



Equalizer from mp3 player

Space-domain: image

- load your image
- display image
- show 2D frequency amplitude, phase, power spectrum, linear/ log
- show rings of equal freq., “image equalizer”
- adjust and display in real time- example on next slide

Fourier Filtering (ImageJ)

