Chapter 9 - Polarization

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Polarization

- The x-y components of the field can be expressed via a 2D "Jones" vector:

$$J = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \end{pmatrix}$$

(9.1)

- Time variation:

$$E_x = A_x \cos(\omega t + \phi_x)$$
$$E_y = A_y \cos(\omega t + \phi_y)$$

(9.2)

Polarization

- Eliminate time:

$$E_x = A_x \left[\cos(\omega t)\cos(\phi_x) - \sin(\omega t)\sin(\phi_x) \right]$$

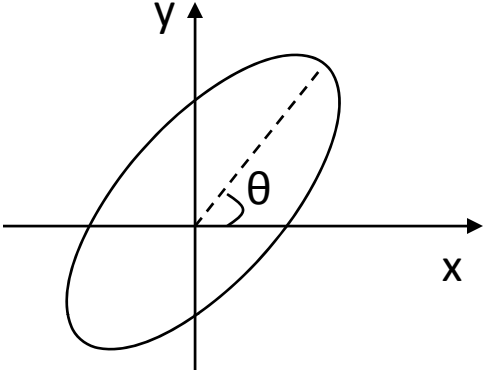$$E_y = A_y \left[\sin(\omega t)\sin(\phi_y) + \cos(\omega t)\cos(\phi_y) \right]$$

(9.3)

- Take as exercise to prove that the trajectory of the E-field is:

$$\frac{E_x^2}{A_x^2} + \frac{E_y^2}{A_y^2} - \frac{2E_x E_y}{A_x A_y}\cos\Delta\phi = \sin^2\Delta\phi$$

(9.4)

where $\Delta\phi = \phi_y - \phi_x$

Polarization

- This is an ellipse:

- In general, it's rotated by angle θ

- This is what we would "see" looking straight at the incoming beam

- There are few particular cases:

a) Linear polarization: $\Delta\phi = 0, \pi$

$$\Rightarrow \left(\frac{E_x}{A_x} \pm \frac{E_y}{A_y} \right)^2 = 0 \Rightarrow \underline{\text{line}}$$
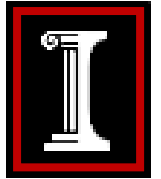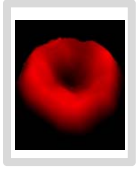
Polarization

b) Circular polarization: $\begin{cases} \Delta\phi = \pm\pi/2 \\ A_x = A_y \end{cases}$ $\begin{aligned} \pi/2 &\to \text{clockwise} \\ -\pi/2 &\to \text{clockwise} \end{aligned}$

$$\Rightarrow E_x{}^2 + E_y{}^2 = A^2$$

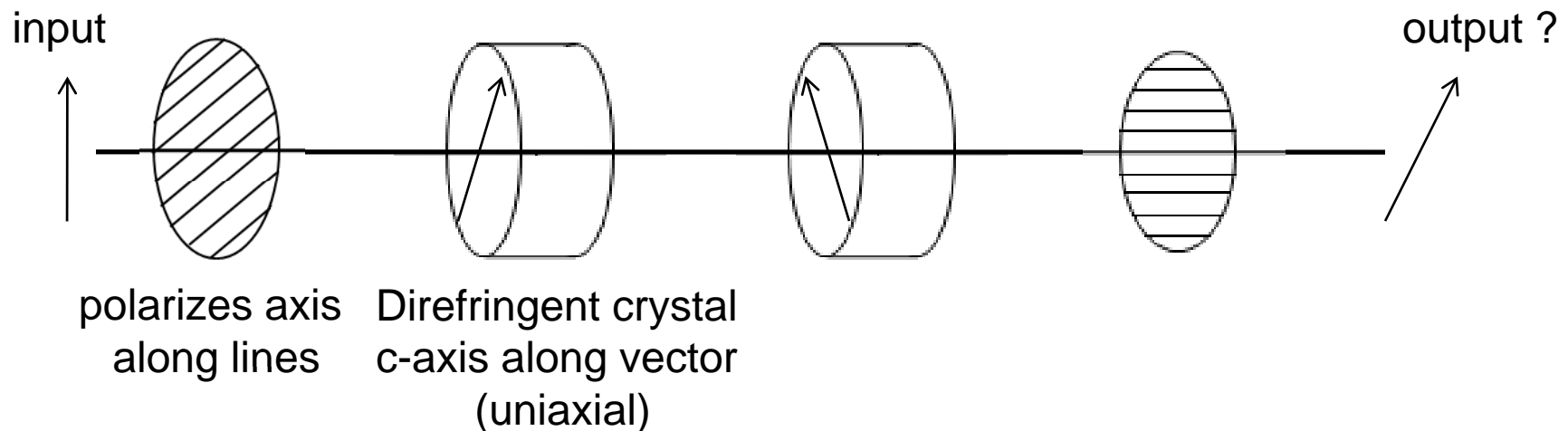c) Straight ellipse: $\begin{cases} \Delta\phi = \pm\pi/2 \\ A_x \neq A_y \end{cases}$

Jones Calculus for
Birefringent Optical Systems

Introduction

- ABCD matrices told us about amplitude distribution on propagation

- Jones "calculus" deals with polarization changes for polarization sensitive optical elements

input   output ?

polarizes axis
along lines

Direfringent crystal
c-axis along vector
(uniaxial)

Introduction

- Jones "calculus" developed around 1940

- Based on 2x2 matrices (similat to ABCD)

  - Need 2 to describe Z polarization

  - Light travels 1 of 2 tranverse normal modes

- Birefringent crystals play role because two modes travel at different phase velocities

- Uniaxials simple – c-axis in plate surface

Retardation Plates (Waveplates)

- Change polarization state

- <u>Assumption</u>: no reflections at surface of elements (can use anti-reflection cratings)

- Use Jones matrices discussed before

$$\overline{E} = \frac{1}{2}\vec{V}\underbrace{E(x, y, z)}_{\text{Slow variation O.K.}}e^{j(\omega t - kz)} + cc$$

describes
polarization

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} ; \quad V_x^2 + V_y^2 = 1$$

Retardation Plates (Waveplates)

- E.g.
  - x-polarized $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

    $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}$

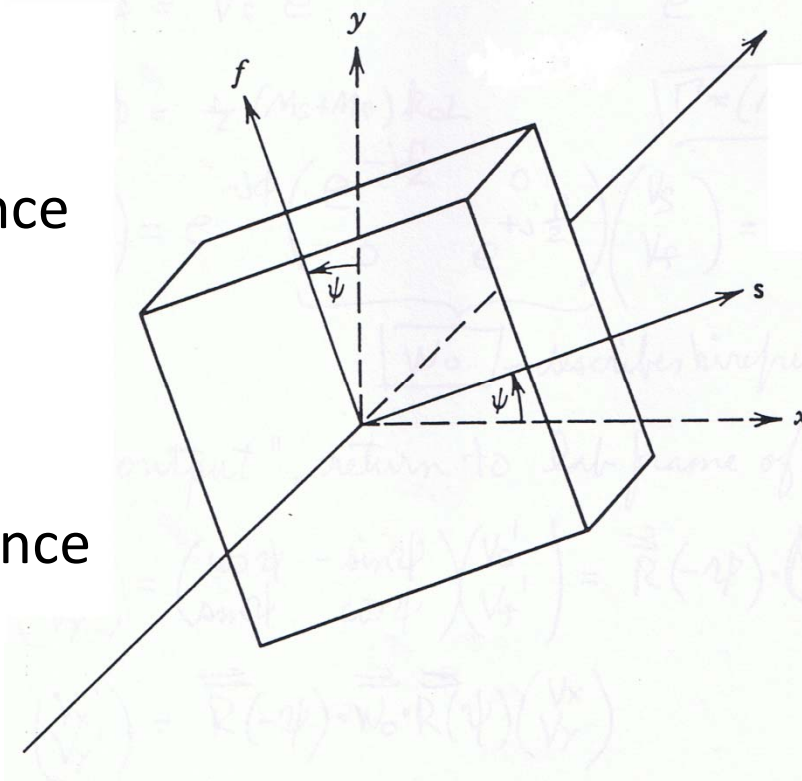 - y-polarized $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

    $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix}$

Retardation Plates (Waveplates)

x,y,z
laboratory
frame of reference

f,s,z
crystal
frame of reference

f – "fast" axis
s – "slow" axis

$$n_s > n_f$$

Retardation Plates (Waveplates)

- From before

$$\begin{pmatrix} V_s \\ V_f \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

$$\downarrow \qquad \boxed{R(\psi)} \qquad \downarrow$$

just inside the crystal $\qquad$ $\boxed{R(\psi)}$ $\qquad$ just outside the crystal

- V_s & V_f propagate independently in the crystal
- After a distance L: $\quad V_s' = V_s e^{-jn_s k_0 L}$

$$V_f' = V_f e^{-jn_f k_0 L}$$

Rewrite: $\quad V_s' = V_s e^{-j\frac{1}{2}(n_s+n_f)k_0 L} e^{-j\frac{1}{2}(n_s-n_f)k_0 L}$

$$V_f' = V_f e^{-j\frac{1}{2}(n_s+n_f)k_0 L} e^{+j\frac{1}{2}(n_s-n_f)k_0 L}$$

Retardation Plates (Waveplates)

- Define $\phi = \dfrac{1}{2}(n_s + n_f)k_0 L$ $\Gamma = (n_s - n_f)k_0 L$

$$\begin{pmatrix} V_s{}' \\ V_f{}' \end{pmatrix} = e^{-j\phi} \begin{pmatrix} e^{-j\frac{\Gamma}{2}} & 0 \\ 0 & e^{+j\frac{\Gamma}{2}} \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix} = e^{-j\phi} \overline{\overline{W_0}} \begin{pmatrix} V_s \\ V_f \end{pmatrix}$$

$\boxed{W_0}$ - describes birefringent medium

- To get "output", return to lab frame of reference

Retardation Plates (Waveplates)

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} V_s{}' \\ V_f{}' \end{pmatrix} = \overline{\overline{R}}(-\psi)\begin{pmatrix} V_s{}' \\ V_f{}' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \overline{\overline{R}}(-\psi)\cdot\overline{\overline{W_0}}\cdot\overline{\overline{R}}(\psi)\begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

order of doing products of matrices

- <u>Note</u>: each matrix is unitary $\left\| R(-\psi) = 1 \right\|$
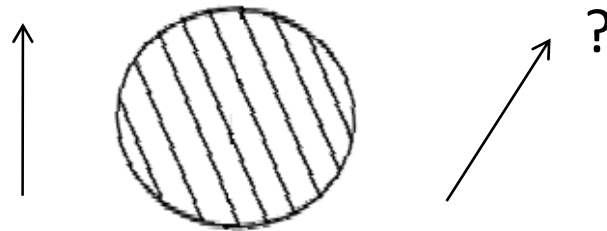 - → Polarization remain orthogonal

Polarizers

x-axis $\quad P_x = e^{-j\phi_p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

y-axis $\quad P_y = e^{-j\phi_p} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

ϕ_p - phase change
      through polarizer

?

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \overline{\overline{R}}(-\psi) \cdot \overline{\overline{P_0}} \cdot \overline{\overline{R}}(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$
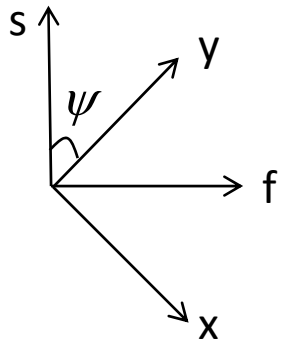
Half (1/2) Wave Plate

$$\boxed{\Gamma = \pi} \Rightarrow L = \frac{\lambda}{2}(n_s - n_f)$$

$$\Rightarrow e^{\pm j\frac{\pi}{2}} \Rightarrow \pm j$$

$$\overline{\overline{W}} = \overline{\overline{R}}(-\psi) \cdot \overline{\overline{W_0}} \cdot \overline{\overline{R}}(\psi)$$

$$= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

$$= j\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} -\cos\psi & -\sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

$$= -j\begin{pmatrix} \cos^2\psi - \sin^2\psi & 2\sin\psi\cos\psi \\ 2\sin\psi\cos\psi & \sin^2\psi - \cos^2\psi \end{pmatrix}$$

½ Wave Plate

$$\overline{\overline{W}} = -j \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}$$

→ rotates a plane of Polarization by 2 x Ψ

↖ angle between lab and crystal frames

- Another important factor is the transmission through the element

$$T = \frac{\left|E'\right|^2}{\left|E\right|^2} = \frac{\left|V_x'\right|^2 + \left|V_y'\right|^2}{\left|V_x\right|^2 + \left|V_y\right|^2}$$

½ Wave Plate

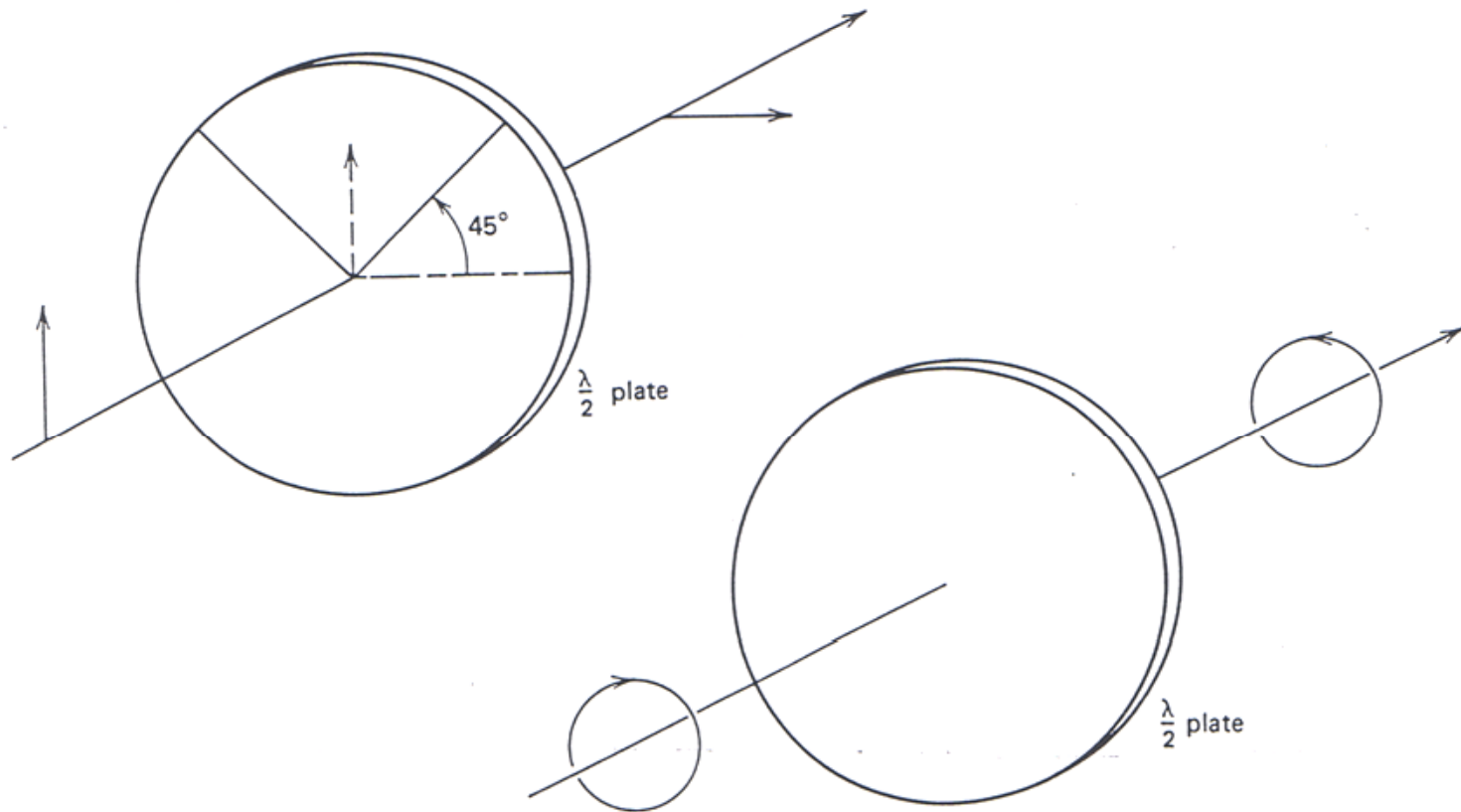- E.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ input

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = -j \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -j \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \end{pmatrix}_{if\,\psi\,=\,45^o} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Just get net polarization rotation

$$\cos^2 2\psi + \sin^2 2\psi = 1 \Rightarrow T = 1$$

½ Wave Plate

½ Wave Plate

- E.g. $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix}$ input

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}\begin{pmatrix} 1 \\ j \end{pmatrix}\frac{1}{\sqrt{2}} = -\frac{j}{\sqrt{2}}\begin{pmatrix} \cos 2\psi + j\sin 2\psi \\ \sin 2\psi - j\cos 2\psi \end{pmatrix}$$

$$T = \frac{1}{2}(\cos 2\psi + j\sin 2\psi)(\cos 2\psi - j\sin 2\psi) + (\sin 2\psi - j\cos 2\psi)(\sin 2\psi + j\cos 2\psi)$$

$$= \frac{1}{2}(\cos^2 2\psi + \sin^2 2\psi + \cos^2 2\psi + \sin^2 2\psi) = 1$$

- Is the output still circulary polarized?
- Depends on Ψ

½ Wave Plate

- Choose $\psi = \dfrac{\pi}{4}$

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = -\frac{j}{\sqrt{2}} \begin{pmatrix} j \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}$$

(1) still circulary polarized

(2) reverses sense of rotation

(3) for $\psi \neq \dfrac{\pi}{4}$, get elliptical polarization!

¼ Wave Plate

$$\boxed{\Gamma = \frac{\pi}{2}} \Rightarrow L = \frac{\lambda}{4}(n_s - n_f)$$

$$\Rightarrow e^{\pm j\frac{\pi}{4}} = \cos\frac{\pi}{4} \pm j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1 \pm j)$$

Assume $\Psi = 45^0$:

$$\overline{\overline{W}} = \overline{\overline{R}}(-45^o) \cdot \overline{\overline{W}}_0\left(\frac{\pi}{2}\right) \cdot \overline{\overline{R}}(45^o) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-j\frac{\pi}{4}} & 0 \\ 0 & e^{j\frac{\pi}{4}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}e^{-j\frac{\pi}{4}}\begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix}$$

¼ Wave Plate
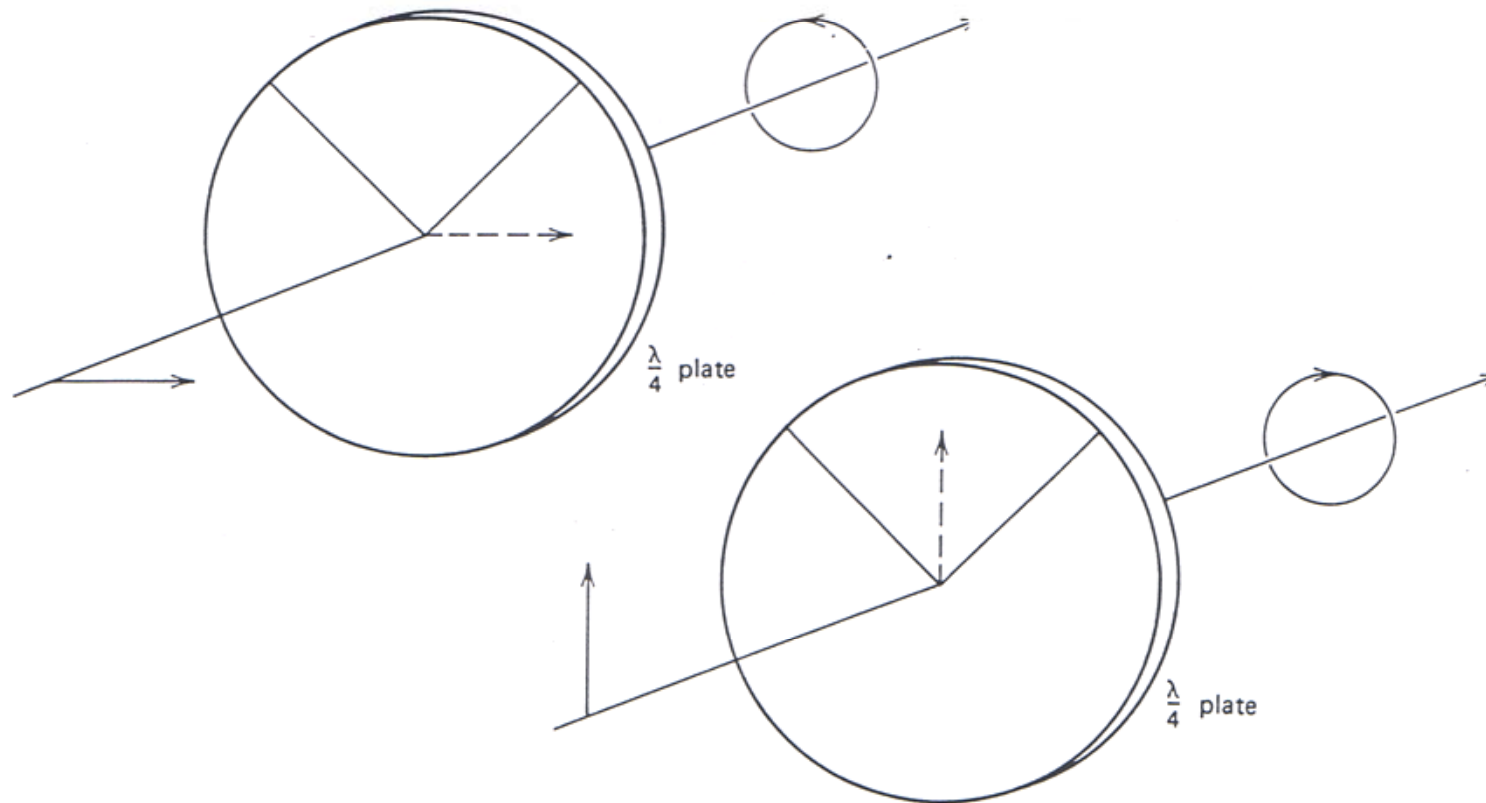
- E.g. $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j \\ 1-j \end{pmatrix}$

$$= \dfrac{1+j}{2} e^{-j\frac{\pi}{4}} \begin{pmatrix} 1 \\ -j \end{pmatrix} \qquad \text{RCP}$$

- Eg. $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ -j \end{pmatrix}; \begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix} \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2\sqrt{2}} \begin{pmatrix} 0 \\ 2-2j \end{pmatrix}$

$$= \dfrac{1-j}{\sqrt{2}} e^{-j\frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{y-polarized}$$

¼ Wave Plate
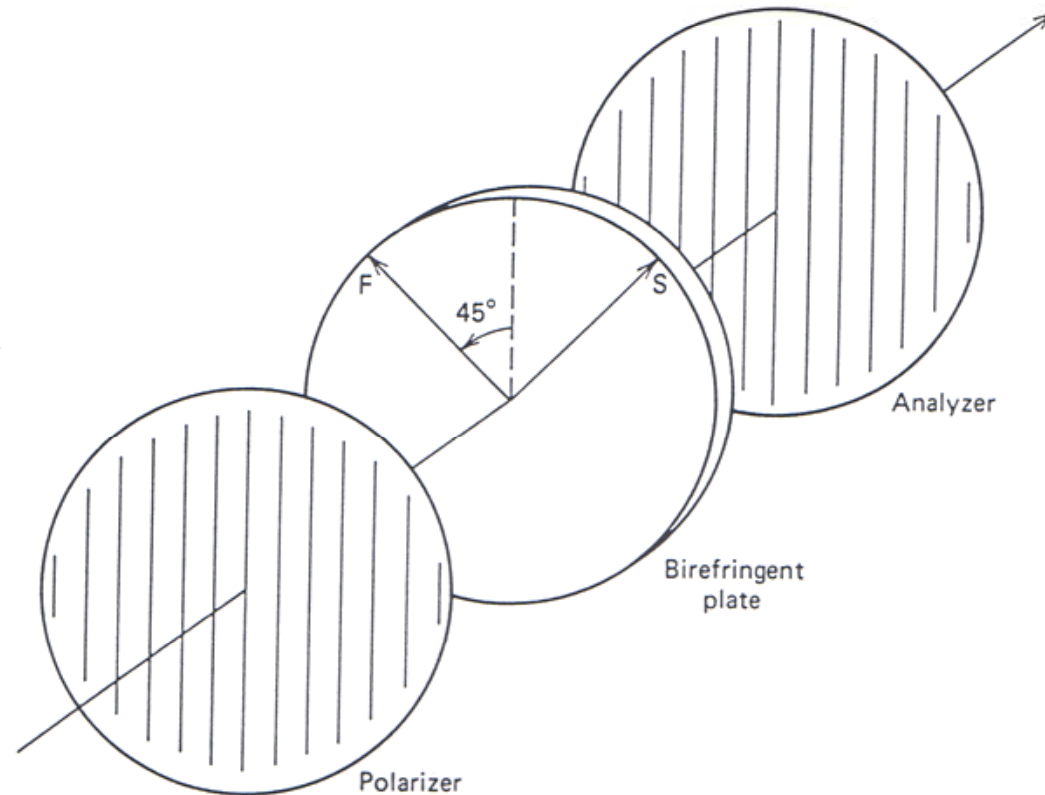
Waveplates and Polarizers

Waveplates and Polarizers

$$W = R(-45^o) \cdot W_0(\Gamma) \cdot R(45^o) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-j\frac{\Gamma}{2}} & 0 \\ 0 & e^{j\frac{\Gamma}{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{2}{2}\begin{pmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{pmatrix}$$

$$Px \cdot W \cdot Px = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ -j\sin\frac{\Gamma}{2} & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

Waveplates and Polarizers

- <u>Case I</u>: unpolarized light $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\dfrac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\Gamma}{2} \\ 0 \end{pmatrix}$$ linearly polarized: obvious !!

$$T = \frac{\left|V_x'\right|^2 + \left|V_y'\right|^2}{\left|V_x\right|^2 + \left|V_y\right|^2} = \frac{1}{2}\cos^2\frac{\Gamma}{2}$$ ½ lost at first polarizer!

Waveplates and Polarizers

- Case II: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\Gamma}{2} \\ 0 \end{pmatrix}$$

$$T = \cos^2\frac{\Gamma}{2}$$

Waveplates and Polarizers

- <u>Case III</u>: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}$

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\Gamma}{2} \\ 0 \end{pmatrix}$$

$$T = \frac{1}{2}\cos^2\frac{\Gamma}{2}$$

- For ½ wave plates, $\Gamma = \pi \Rightarrow T = 0$

Birefringent Plate between Crossed Polarizers

$\Psi = 45^0$:

$$W = R(-45^o) \cdot W_0(\Gamma) \cdot R(45^o) = \begin{pmatrix} \cos\dfrac{\Gamma}{2} & -j\sin\dfrac{\Gamma}{2} \\ -j\sin\dfrac{\Gamma}{2} & \cos\dfrac{\Gamma}{2} \end{pmatrix}$$

$$Py \cdot W \cdot Px = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\dfrac{\Gamma}{2} & -j\sin\dfrac{\Gamma}{2} \\ -j\sin\dfrac{\Gamma}{2} & \cos\dfrac{\Gamma}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -j\sin\dfrac{\Gamma}{2} & 0 \end{pmatrix}$$

Birefringent Plate between Crossed Polarizers

- Case I: unpolarized light $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -j\sin\frac{\Gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -j\sin\frac{\Gamma}{2} \end{pmatrix}$$

$$T = \frac{1}{2}\sin^2\frac{\Gamma}{2}$$

Birefringent Plate between Crossed Polarizers

- Case II: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} V_x{'} \\ V_y{'} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -j\sin\dfrac{\Gamma}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -j\sin\dfrac{\Gamma}{2} \end{pmatrix}$$

$$T = \sin^2\frac{\Gamma}{2}$$