ECE-656: Fall 2011

Lecture 19:

Scattering I:
Collision Integral

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characteristic times

1) single particle lifetime, τ :

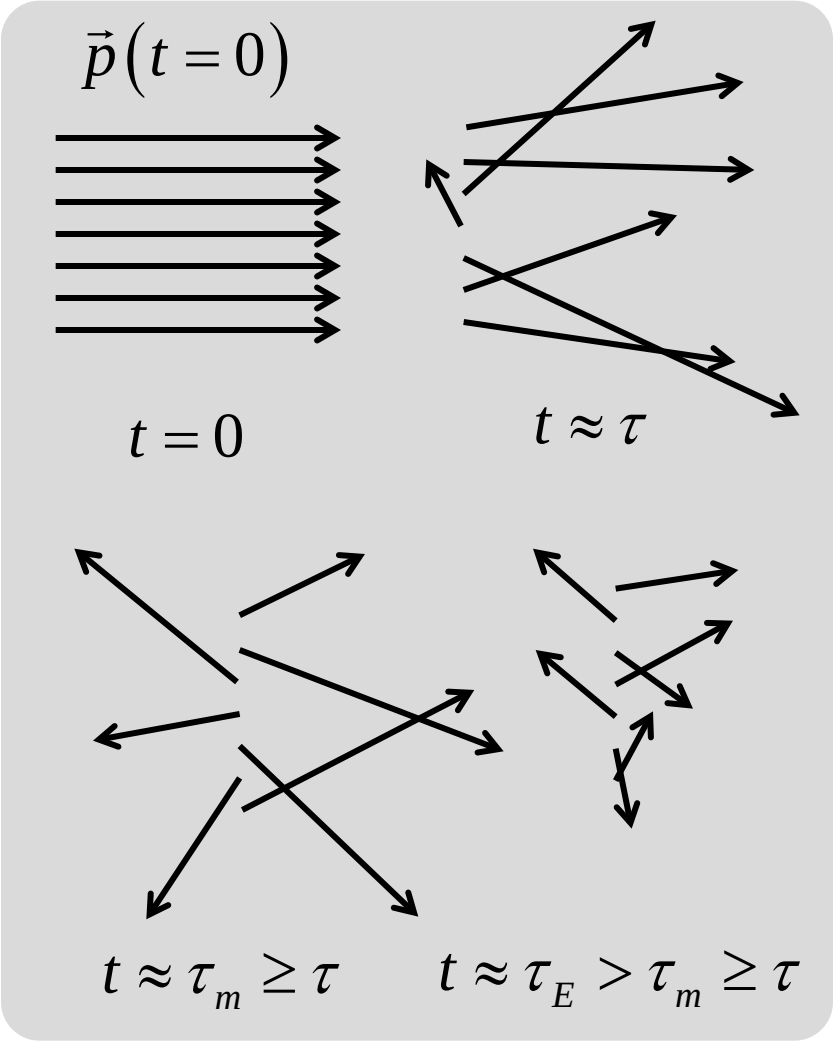
$$\tau\left(\vec{p}\right)$$

2) momentum relaxation time, τ_m :

$$\tau_m\left(\vec{p}\right)$$

3) energy relaxation time, τ_E :

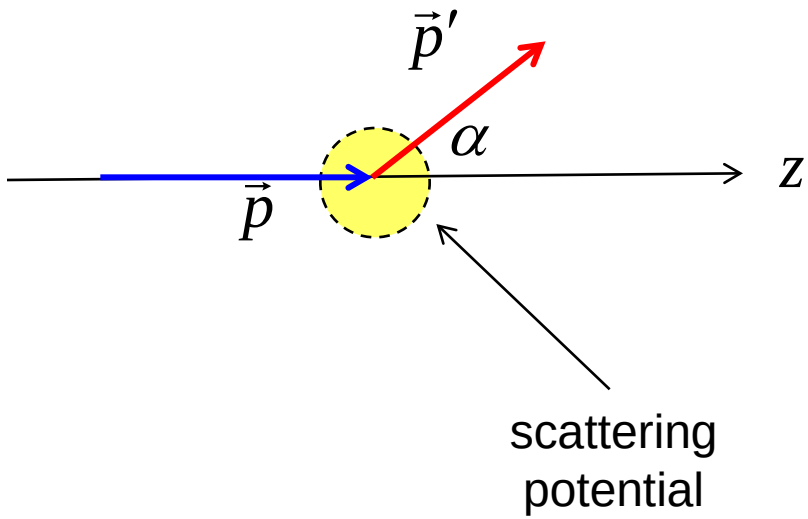
$$\tau_E\left(\vec{p}\right)$$



transition rate

Transition rate from *p* to *p'* (probability per second)

$$S\left(\vec{p} \rightarrow \vec{p}'\right)$$

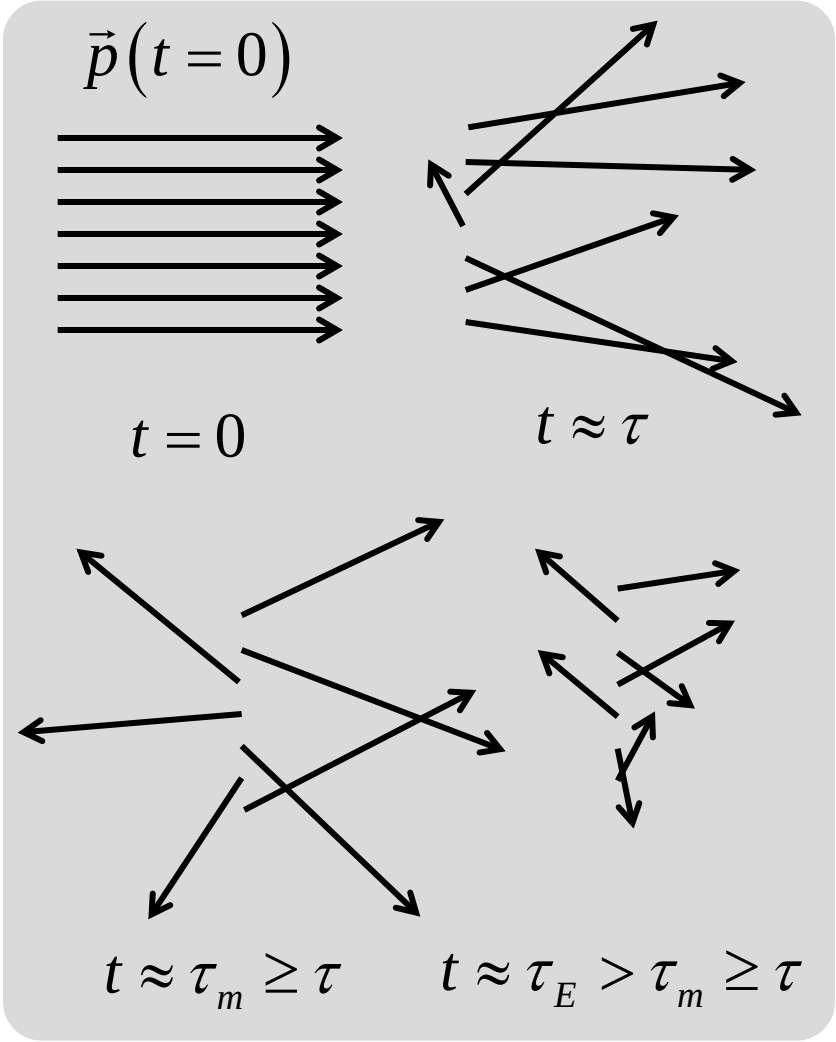


characteristic times

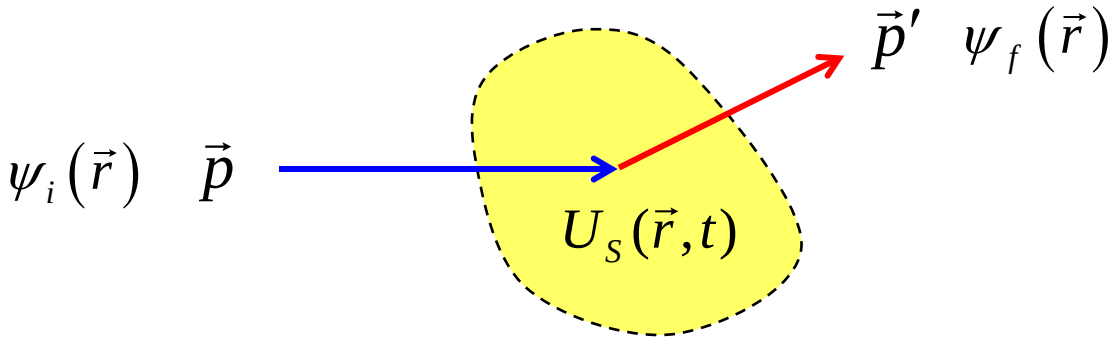
$$\frac{1}{\tau(\vec{p})} = \sum_{\vec{p}',\uparrow} S(\vec{p} \to \vec{p}')$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{\vec{p}',\uparrow} S(\vec{p} \to \vec{p}') \frac{\Delta p_z}{p_z}$$

$$\frac{1}{\tau_E(\vec{p})} = \sum_{\vec{p}',\uparrow} S(\vec{p} \to \vec{p}') \frac{\Delta E}{E_0}$$



Fermi's Golden Rule



$$S(\vec{p} \to \vec{p}') = \frac{2\pi}{\hbar}\left|H_{p'p}\right|^2 \delta\left(E' - E - \Delta E\right) \qquad H_{\vec{p}',\vec{p}} = \int_{-\infty}^{+\infty} \psi_f^* U_S(\vec{r})\psi_i d\vec{r}$$

To be discussed in Lecture 21

the Boltzmann Transport Equation

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = \hat{C} f$$

$$\hat{C} f\left(\vec{r}, \vec{p}, t\right) = \sum_{p'} S\left(\vec{p}', \vec{p}\right) f\left(\vec{p}'\right)\left[1 - f\left(\vec{p}\right)\right] - \sum_{p'} S\left(\vec{p}, \vec{p}'\right) f\left(\vec{p}\right)\left[1 - f\left(\vec{p}'\right)\right]$$

scattering operator

$$\left.\frac{df}{dt}\right|_{coll} = \hat{C}f\left(\vec{r},\vec{p},t\right) = \text{in-scattering rate - out-scattering rate}$$

$$\text{in-scattering rate} = \sum_{p'} S\left(\vec{p}' \to \vec{p}\right) f\left(\vec{p}'\right)\left[1 - f\left(\vec{p}\right)\right]$$

$$\text{out-scattering rate} = \sum_{p'} S\left(\vec{p} \to \vec{p}'\right) f\left(\vec{p}\right)\left[1 - f\left(\vec{p}'\right)\right]$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = \sum_{p'} S\left(\vec{p}' \to \vec{p}\right) f\left(\vec{p}'\right)\left[1 - f\left(\vec{p}\right)\right] - \sum_{p'} S\left(\vec{p} \to \vec{p}'\right) f\left(\vec{p}\right)\left[1 - f\left(\vec{p}'\right)\right]$$

nondegenerate scattering operator

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}'\rightarrow\vec{p}\right)f\left(\vec{p}'\right)\left[1-f\left(\vec{p}\right)\right]-\sum_{p'}S\left(\vec{p}\rightarrow\vec{p}'\right)f\left(\vec{p}\right)\left[1-f\left(\vec{p}'\right)\right]$$

probability that the state at p' is occupied

probability that the state at p is empty

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}'\rightarrow\vec{p}\right)f\left(\vec{p}'\right)-\sum_{p'}S\left(\vec{p}\rightarrow\vec{p}'\right)f\left(\vec{p}\right)$$

non-degenerate scattering operator
(assumes final state empty)

outline

1. Review
2. **Collision operator**
3. Electron-electron scattering
4. Discussion
5. Summary

conservation of carriers

We are discussing scattering mechanisms that move carriers around in *k*-space. They do not create or destroy carriers.

$$\sum_{p} \hat{C} f\left(\vec{r}, \vec{p}, t\right) = 0$$

$$\sum_{p} \left\{ \sum_{p'} S\left(\vec{p}', \vec{p}\right) f\left(\vec{p}'\right) - \sum_{p'} S\left(\vec{p}, \vec{p}'\right) f\left(\vec{p}\right) \right\} = 0$$

$$\sum_{p,p'} S\left(\vec{p}' \rightarrow \vec{p}\right) f\left(\vec{p}'\right) = \sum_{p,p'} S\left(\vec{p} \rightarrow \vec{p}'\right) f\left(\vec{p}\right) \quad ?$$

conservation of carriers

$$\sum_{p,p'} S\left(\vec{p}' \rightarrow \vec{p}\right) f\left(\vec{p}'\right) = \sum_{p,p'} S\left(\vec{p} \rightarrow \vec{p}'\right) f\left(\vec{p}\right) \quad ?$$

Work on LHS:

$$\sum_{p,p'} S\left(\vec{p}' \rightarrow \vec{p}\right) f\left(\vec{p}'\right) = \sum_{p',p} S\left(\vec{p}' \rightarrow \vec{p}\right) f\left(\vec{p}'\right)$$ (interchange order of summation)

$$\sum_{p,p'} S\left(\vec{p}' \rightarrow \vec{p}\right) f\left(\vec{p}'\right) = \sum_{p,p'} S\left(\vec{p} \rightarrow \vec{p}'\right) f\left(\vec{p}\right)$$ (interchange dummy indices)

QED

physical constraints on scattering

We will be working out the transition rates for various types of scattering processes. What can we say in advance about the physical properties that the resulting transition rates must satisfy?

$$\hat{C}f = \sum_{\mathbf{p}'} S\left(\vec{p}' \to \vec{p}\right) f\left(\vec{p}'\right)\left[1 - f\left(\vec{p}\right)\right] - S\left(\vec{p} \to \vec{p}'\right) f\left(\vec{p}\right)\left[1 - f\left(\vec{p}'\right)\right]$$

in equilibrium: $f\left(\vec{p}\right) = f\left(\vec{p}'\right) = f_0\left(E\right)$

and the principle of detailed balance holds.

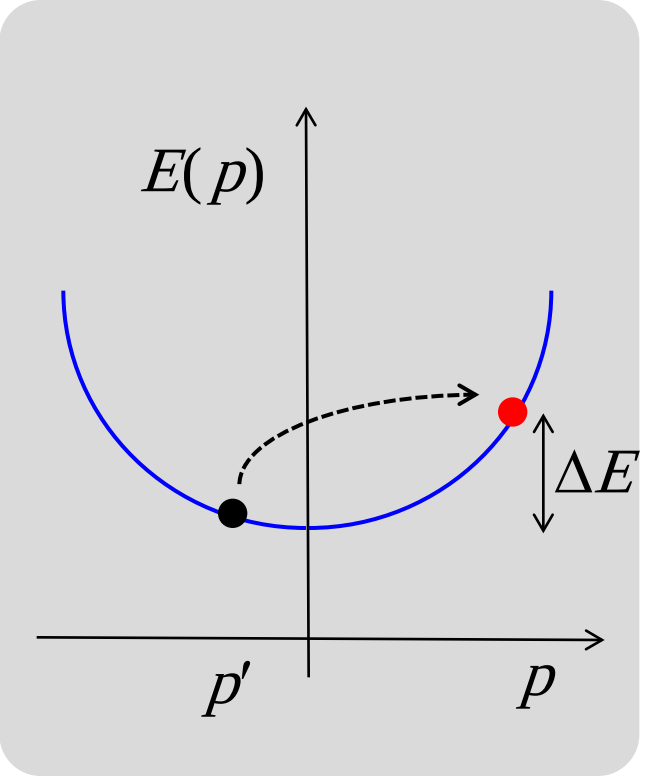
detailed balance in equilibrium

$$S_0\left(\vec{p}' \to \vec{p}\right) f_0\left(E'\right)\left[1 - f_0\left(E\right)\right] - S_0\left(\vec{p} \to \vec{p}'\right) f_0\left(E\right)\left[1 - f\left(E'\right)\right] = 0$$

holds out of equilibrium too….

$$\frac{S\left(\vec{p}' \to \vec{p}\right)}{S\left(\vec{p} \to \vec{p}'\right)} = \frac{f_0\left(E\right)\left[1 - f_0\left(E'\right)\right]}{f_0\left(E'\right)\left[1 - f_0\left(E\right)\right]} = e^{-\Delta E/k_B T}$$

$$\Delta E = E(p) - E(p')$$



elastic vs. inelastic scattering

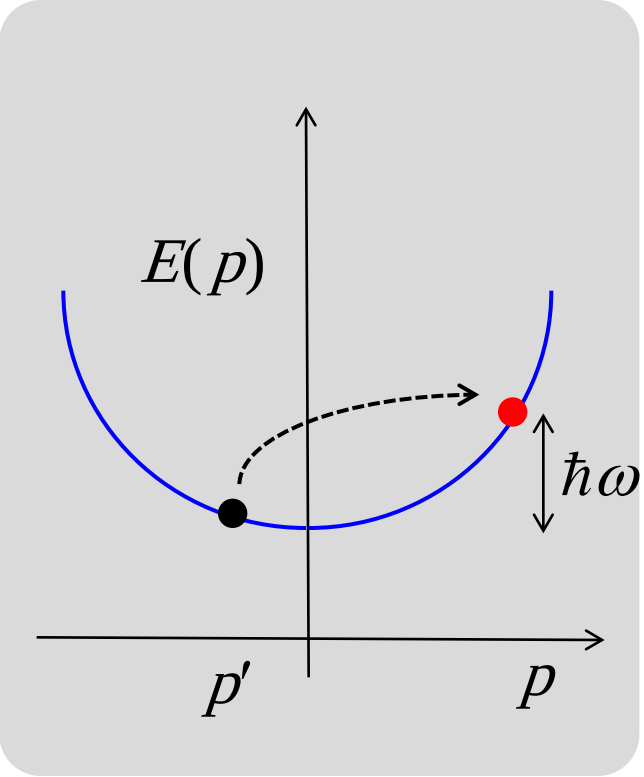
$$\frac{S\left(\vec{p}' \to \vec{p}\right)}{S\left(\vec{p} \to \vec{p}'\right)} = e^{-\Delta E/k_B T} \qquad \Delta E = E(p) - E(p')$$

1) elastic scattering:

$$S\left(\vec{p} \to \vec{p}'\right) = S\left(\vec{p}' \to \vec{p}\right)$$

2) phonon scattering:

$$\frac{S^{ABS}\left(\vec{p}' \to \vec{p}\right)}{S^{EMS}\left(\vec{p} \to \vec{p}'\right)} = e^{-\hbar\omega/k_B T}$$



constraints on phonon scattering

$$\frac{S^{ABS}\left(\vec{p}' \rightarrow \vec{p}\right)}{S^{EMS}\left(\vec{p} \rightarrow \vec{p}'\right)} = e^{-\hbar\omega/k_B T}$$

$$S^{ABS} \sim n_0\left(\hbar\omega\right)$$

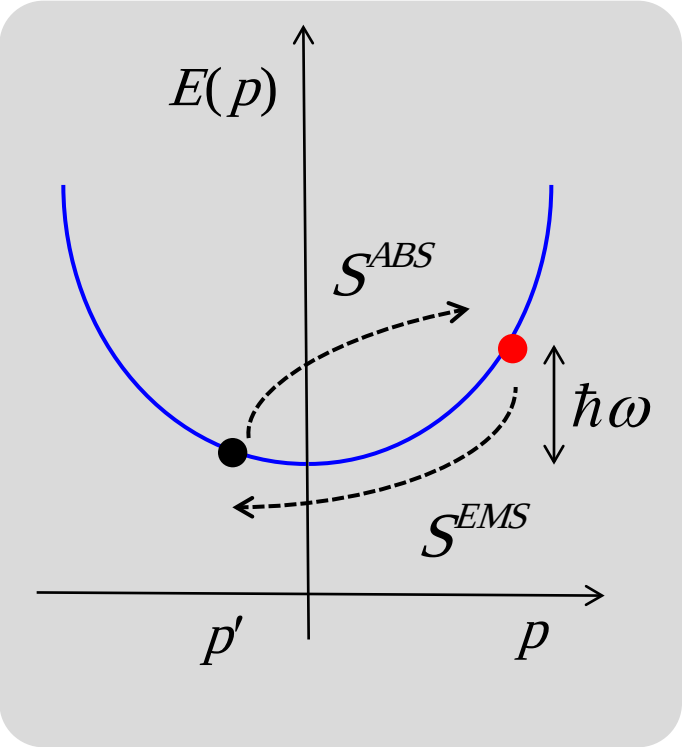
$$S^{EMS} \sim e^{\hbar\omega/k_B T} n_0\left(\hbar\omega\right)$$

$$n_0\left(\hbar\omega\right) = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$$e^{\hbar\omega/k_B T} n_0\left(\hbar\omega\right) = \frac{e^{\hbar\omega/k_B T}}{e^{\hbar\omega/k_B T} - 1} = n_0\left(\hbar\omega\right) + 1$$

S^{ABS} : n_0

S^{EMS} : $n_0 + 1$



outline

1. Review
2. Collision operator
3. **Electron-electron scattering**
4. Discussion
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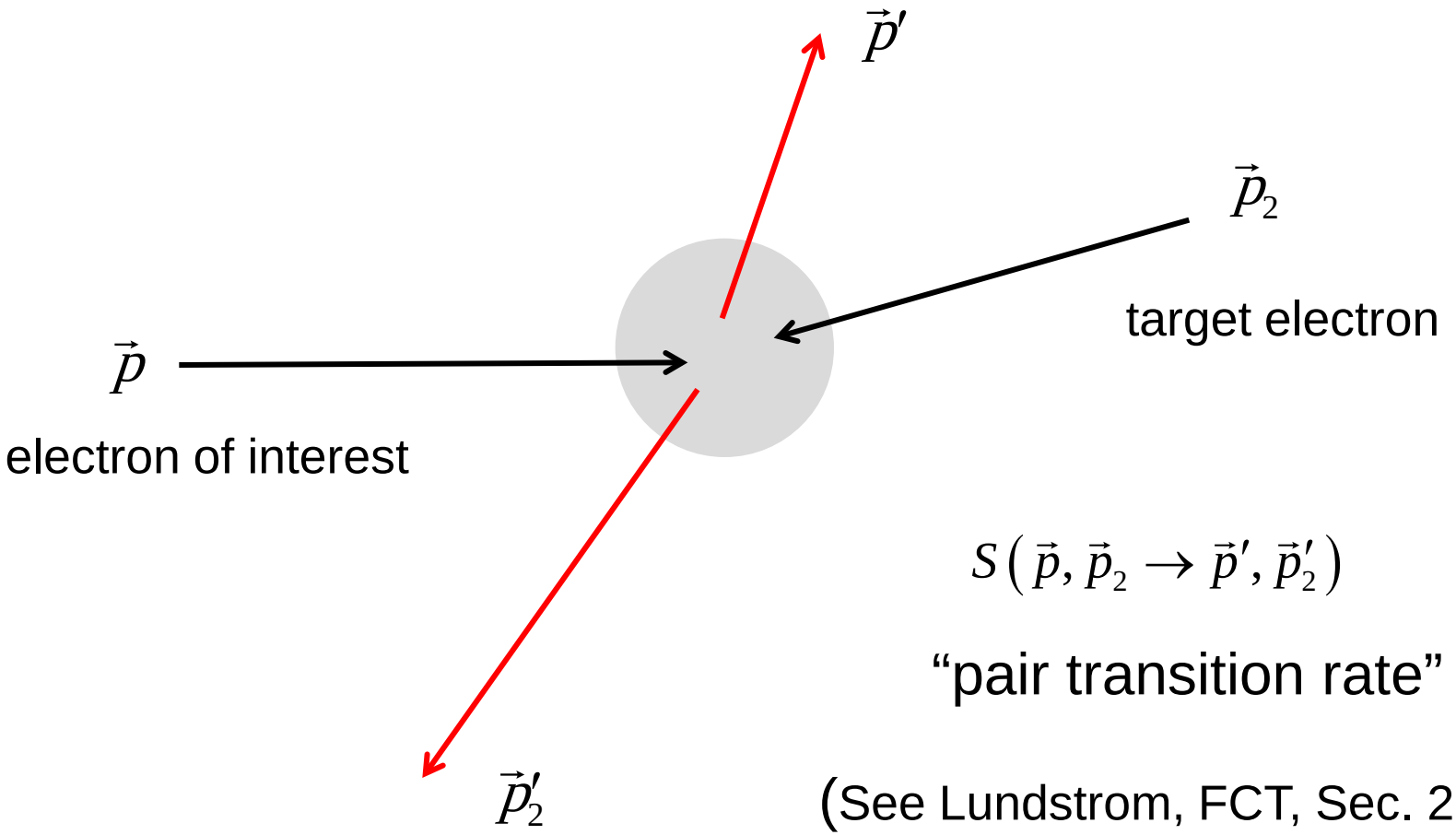
binary collisions

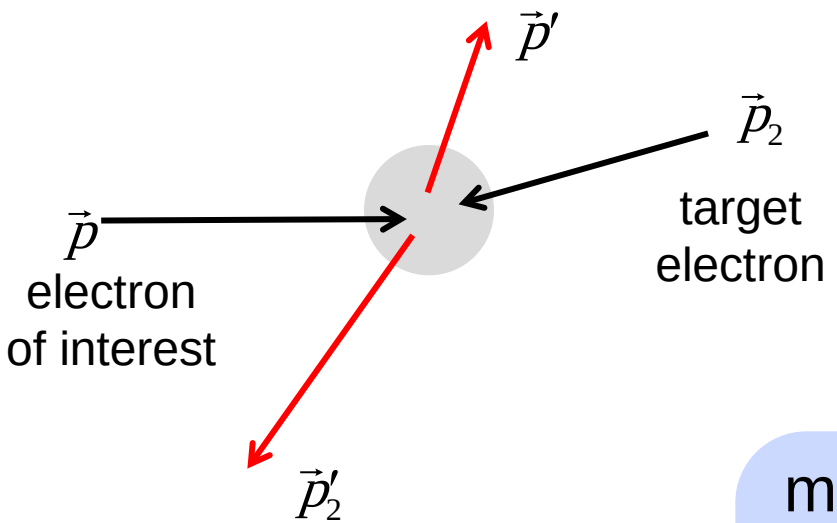


$$S\left(\vec{p}, \vec{p}_2 \rightarrow \vec{p}', \vec{p}_2'\right)$$

"pair transition rate"

(See Lundstrom, FCT, Sec. 2.10)

energy-momentum conservation



$$S\left(\vec{p}, \vec{p}_2 \rightarrow \vec{p}', \vec{p}_2'\right)$$

"pair transition rate"

momentum conservation:

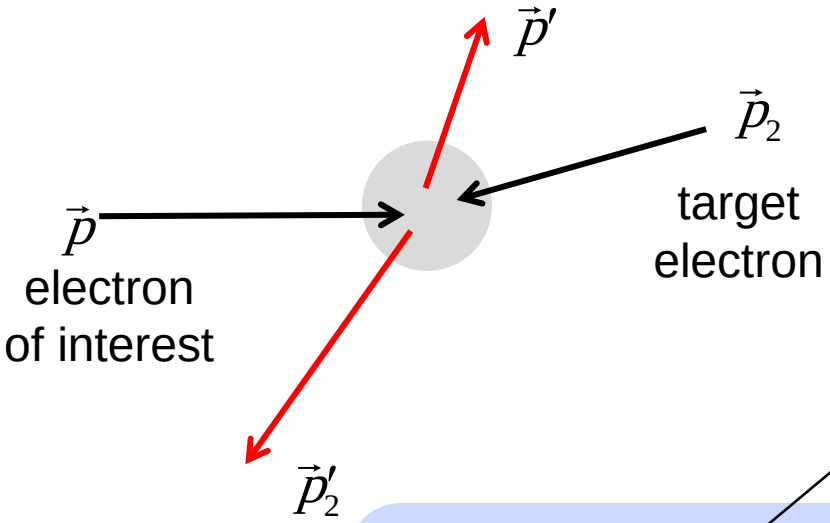
$$\vec{p} + \vec{p}_2 = \vec{p}' + \vec{p}_2'$$

energy conservation:

$$E + E_2 = E' + E_2'$$

(See Lundstrom, FCT, Sec. 2.10)

scattering rate



$$S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right)$$

"pair transition rate"

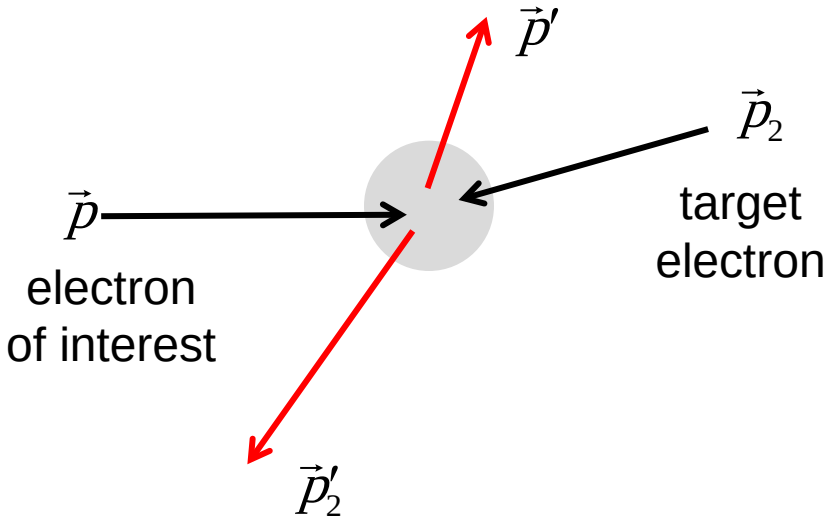
(no sum over p_2')

e-e scattering rate:

$$\frac{1}{\tau_{ee}\left(\vec{p}\right)} = \sum_{p_2}\sum_{p'} S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right) f\left(\vec{p}_2\right)\left[1-f\left(\vec{p}'\right)\right]\left[1-f\left(\vec{p}_2'\right)\right]$$

But, we often argue that this type of scattering has little effect. Why?

collision operator for e-e scattering



$$S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right)$$

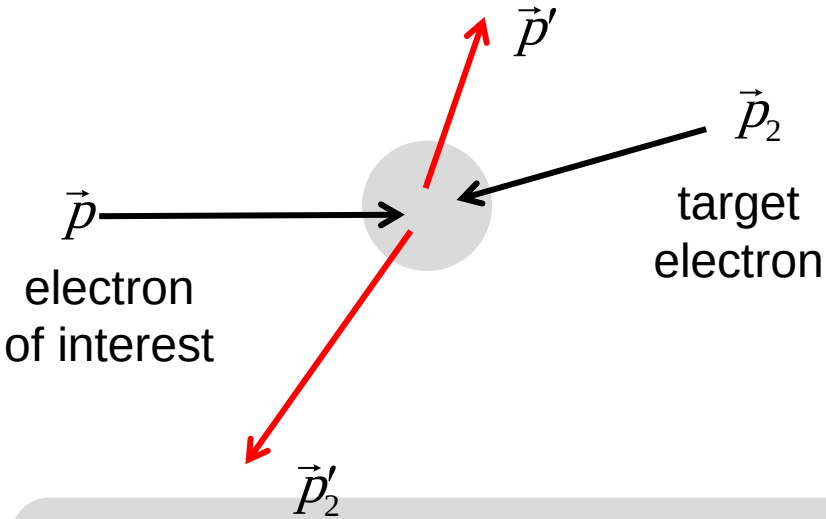
"pair transition rate"

$$\hat{C}f\left(\vec{p}\right)=\sum_{p',p_2'} S\left(\vec{p}',\vec{p}_2' \rightarrow \vec{p},\vec{p}_2\right)f\left(\vec{p}'\right)f\left(\vec{p}_2'\right)-\sum_{p',p_2} S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right)f\left(\vec{p}\right)f\left(\vec{p}_2\right)$$

(assumes MB statistics)

(See Lundstrom, FCT, Sec. 3.3)

collision operator for e-e scattering



$$S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right)$$

"pair transition rate"

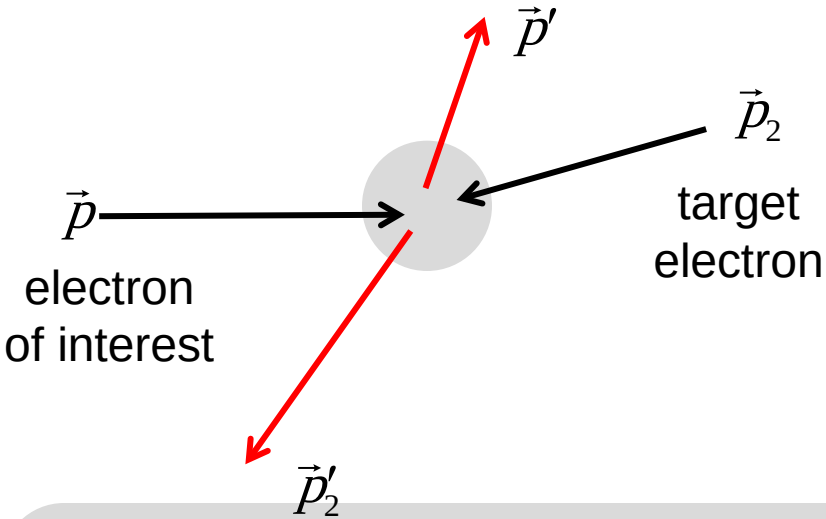
$$\hat{C}f\left(\vec{p}\right)=\sum_{p',p_2'} S\left(\vec{p}',\vec{p}_2' \rightarrow \vec{p},\vec{p}_2\right) f\left(\vec{p}'\right) f\left(\vec{p}_2'\right)-\sum_{p',p_2} S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right) f\left(\vec{p}\right) f\left(\vec{p}_2\right)$$

For e-e scattering, "it can be shown"

$$S\left(\vec{p},\vec{p}_2 \rightarrow \vec{p}',\vec{p}_2'\right)=S\left(\vec{p}',\vec{p}_2' \rightarrow \vec{p},\vec{p}_2\right)$$

(See Lundstrom, FCT, Sec. 3.3)

collision operator for e-e scattering



$$S\left(\vec{p},\vec{p}_2 \to \vec{p}',\vec{p}_2'\right)$$

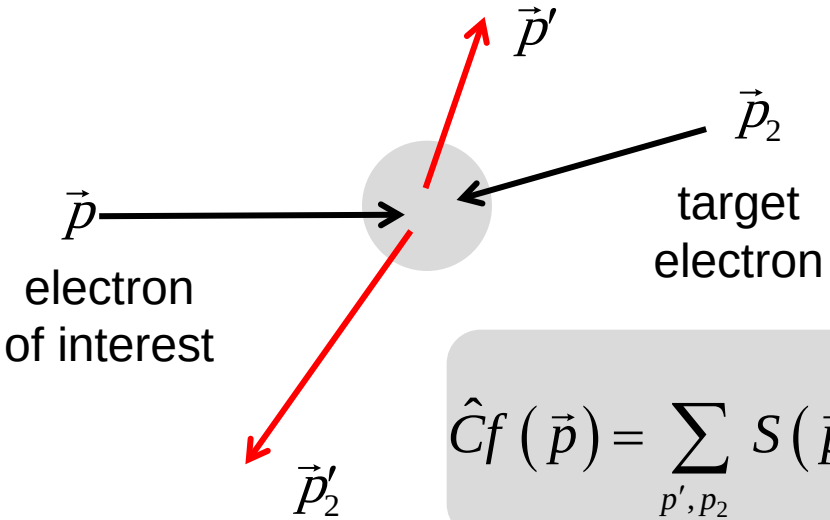
"pair transition rate"

$$\hat{C}f\left(\vec{p}\right) = \sum_{p',p_2'} S\left(\vec{p}',\vec{p}_2' \to \vec{p},\vec{p}_2\right) f\left(\vec{p}'\right) f\left(\vec{p}_2'\right) - \sum_{p',p_2} S\left(\vec{p},\vec{p}_2 \to \vec{p}',\vec{p}_2'\right) f\left(\vec{p}\right) f\left(\vec{p}_2\right)$$

$$S\left(\vec{p},\vec{p}_2 \to \vec{p}',\vec{p}_2'\right) = S\left(\vec{p}',\vec{p}_2' \to \vec{p},\vec{p}_2\right)$$

$$\hat{C}f\left(\vec{p}\right) = \sum_{p',p_2} S\left(\vec{p},\vec{p}_2 \to \vec{p}',\vec{p}_2'\right)\left[f\left(\vec{p}\right) f\left(\vec{p}_2\right) - f\left(\vec{p}'\right) f\left(\vec{p}_2'\right)\right]$$

collision operator for e-e scattering



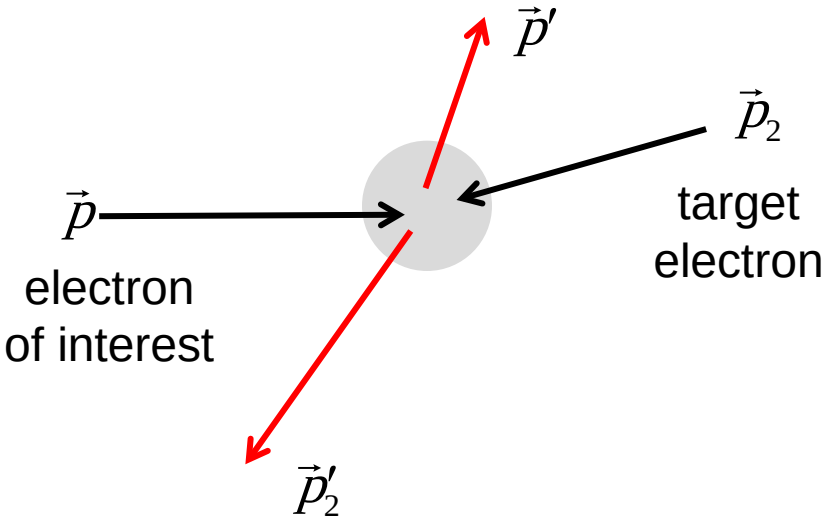
$$\hat{C}f(\vec{p}) = \sum_{p',p_2} S(\vec{p},\vec{p}_2 \to \vec{p}',\vec{p}_2')\left[f(\vec{p})f(\vec{p}_2) - f(\vec{p}')f(\vec{p}_2')\right]$$

Eventually, in-scattering and out-scattering balance and steady-state conditions with no change in *f* occur.

$$\hat{C}f(\vec{p}) = 0 \quad f(\vec{p})f(\vec{p}_2) = f(\vec{p}')f(\vec{p}_2')$$

What distribution function satisfies this equation?

collision operator for e-e scattering



$$\hat{C}f(\vec{p}) = 0 \rightarrow f(\vec{p})\, f(\vec{p}_2) = f(\vec{p}')\, f(\vec{p}_2')$$

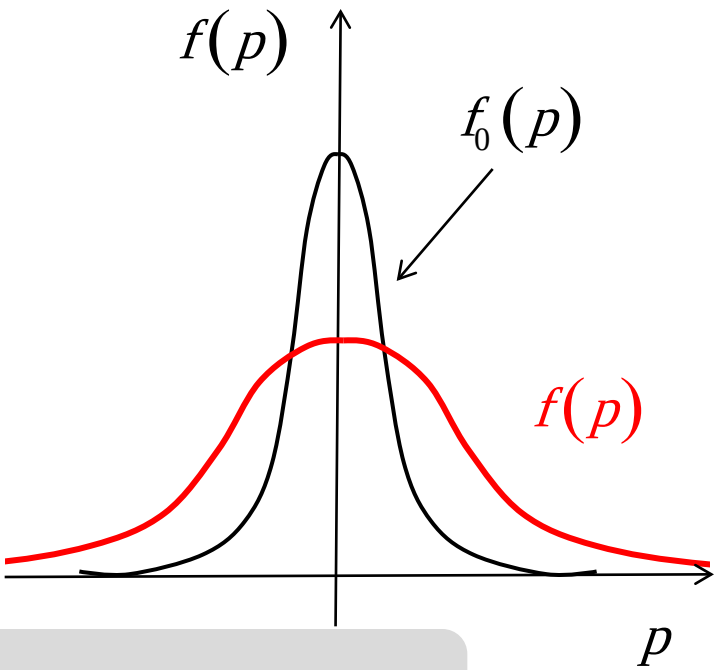
Try a solution of the form: $f(\vec{p}) = e^{C_1 E}$ $\qquad e^{C_1(E+E_2)} = e^{C_1(E'+E_2')}$

If we set the energy to $3/2 k_B T_e$, we find:

$$f(\vec{p}) = e^{-E/k_B T_e} = e^{-p^2/2m^* k_B T_e}$$

Electron-electron scattering produces a Maxwellian distribution.

e-e scattering



$$f_0(p) = e^{p^2/2m^* k_B T_L}$$

$$f_S(p) = e^{p^2/2m^* k_B T_e}$$

$$T_e > T_L$$

- strong e-e scattering produces a Maxwellian or FD distribution.
- To first order, e-e scattering has no effect, because energy and momentum are conserved.
- but it can have non-negligible effects.
- e-e scattering typicall dominates when $n > 10^{17} - 10^{18}$ cm^{-3}.

outline

1. Review
2. Collision operator
3. Electron-electron scattering
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HW prob. 3.12 (FCT)

3.12. Assume that f(p) = $f_S(p)$ is a Fermi-Dirac distribution at the lattice temperature, but don't assume that $f_0 = f_S$. Show that $df/dt|_{coll} = 0$.

$$f_0(p) = \frac{1}{1+e^{(E-E_F)/k_B T_L}} \qquad f_S(p) = \frac{1}{1+e^{(E-F_n)/k_B T_L}}$$

$$f(p) = f_0(p) + \delta f(p) \qquad f(p) = f_S(p) + \delta f(p)$$

$$\hat{C}f(p) \approx \frac{-\delta f(p)}{\tau_m} \qquad \hat{C}f(p) \approx \frac{-\delta f(p)}{\tau_m} \quad ?$$

$$\hat{C}f_0(p) = 0 \qquad \hat{C}f_S(p) = 0\ ?$$

HW prob. 3.12 (FCT)

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}',\vec{p}\right)f\left(\vec{p}'\right)\left[1-f\left(\vec{p}\right)\right]-\sum_{p'}S\left(\vec{p},\vec{p}'\right)f\left(\vec{p}\right)\left[1-f\left(\vec{p}'\right)\right]$$

$$S\left(\vec{p}',\vec{p}\right)f_0\left(\vec{p}'\right)\left[1-f_0\left(\vec{p}\right)\right]=S\left(\vec{p},\vec{p}'\right)f_0\left(\vec{p}\right)\left[1-f_0\left(\vec{p}'\right)\right]$$

$$S\left(\vec{p}',\vec{p}\right)f_0\left(\vec{p}'\right)=S\left(\vec{p},\vec{p}'\right)f_0\left(\vec{p}\right)$$ (Maxwell-Botzmann)

$$\begin{aligned}S\left(\vec{p}',\vec{p}\right)f_S\left(\vec{p}'\right)&=S\left(\vec{p}',\vec{p}\right)e^{\left(F_n-E_C\right)/k_BT_L}\\&=e^{\left(F_n-E_F\right)/k_BT_L}S\left(\vec{p}',\vec{p}\right)e^{\left(E_F-E_C\right)/k_BT_L}\\&=e^{\left(F_n-E_F\right)/k_BT_L}S\left(\vec{p}',\vec{p}\right)f_0\left(\vec{p}\right)\end{aligned}$$

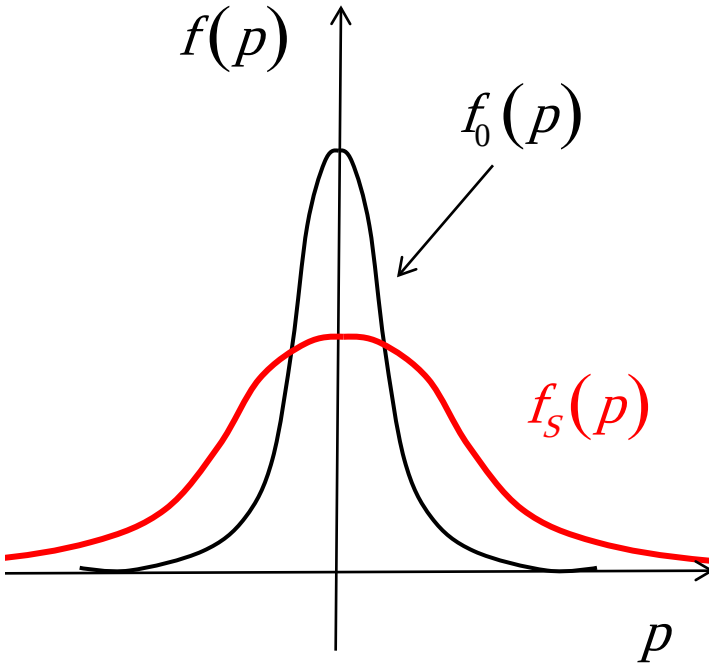
HW prob. 3.12 (FCT)

$$S(\vec{p}',\vec{p})\, f_0(\vec{p}') = S(\vec{p},\vec{p}')\, f_0(\vec{p})$$ (Maxwell-Botzmann)

$$S(\vec{p}',\vec{p})\, f_S(\vec{p}') = e^{(F_n - E_F)/k_B T_L}\, S(\vec{p}',\vec{p})\, f_0(\vec{p})$$

$$S(\vec{p},\vec{p}')\, f_S(\vec{p}') = e^{(F_n - E_F)/k_B T_L}\, S(\vec{p},\vec{p}')\, f_0(\vec{p})$$

$$S(\vec{p}',\vec{p})\, f_S(\vec{p}') = S(\vec{p},\vec{p}')\, f_S(\vec{p})$$



$$f_S(p) \propto e^{p^2/2m^* k_B T_e}$$

So it works I this case, but, it does not work when $T_e \neq T_L$

outline

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summary

1) The transition rate, S($p \rightarrow p'$) determines the characteristics times.

2) The collision operator conserves particles.

3) Phonon absorption is proportional to n, emission to n+1.

4) Electron-electron scattering leads to Maxwellian (Fermi-Dirac) distributions.

questions

1. Review
2. Collision operator
3. Electron-electron scattering
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