

ECE-656: Fall 2011

Lecture 20:

**Scattering II:
Relaxation time approximation**

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outline

- 1. Justification of the RTA**
2. Discussion
3. HW prob. 17



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the BTE and the RTA

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \vec{F}_e \cdot \nabla_p f = \hat{C} f$$

$$\hat{C} f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') [1 - f(\vec{p})] - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p}) [1 - f(\vec{p}')]]$$

$$\hat{C} f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p}) \quad (\text{Boltzmann statistics})$$

$$\hat{C} f(\vec{r}, \vec{p}, t) = -\frac{(f - f_s)}{\tau_f(\vec{r}, \vec{p})}$$

1) Under what conditions can we approximate the collision integral with the RTA?

2) When we can, how is τ_f defined?

symmetric and antisymmetric components

$$f(\vec{p}) = f_s(\vec{p}) + \delta f(\vec{p})$$

$$f_s(\vec{p}) = f_s(-\vec{p}) \quad \text{symmetric in momentum}$$

$$\delta f(\vec{p}) = -\delta f(-\vec{p}) \quad \text{anti-symmetric in momentum}$$

$$f(\vec{p}) = f_s(\vec{p}) + f_A(\vec{p})$$

simplified collision integral

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p}) \quad (\text{non-degenerate})$$

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) [f_S(\vec{p}') + f_A(\vec{p}')] - \sum_{p'} S(\vec{p}, \vec{p}') [f_S(\vec{p}) + f_A(\vec{p})]$$

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_S(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_S(\vec{p}) = 0 \quad (\text{equilibrium})$$

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_A(\vec{p}) \quad (\text{assumes near-equilibrium – and MB statistics})$$

the “equilibrium simplification”

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p})$$

(return to the first form of the equation)

Can we relate $S(\vec{p}', \vec{p})$ and $S(\vec{p}, \vec{p}')$?

Yes, in equilibrium.

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_0(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_0(\vec{p}) = 0$$

(equilibrium)

$$S(\vec{p}', \vec{p}) f_0(\vec{p}') - S(\vec{p}, \vec{p}') f_0(\vec{p}) = 0$$

(detailed balance)

$$S(\vec{p}', \vec{p}) = S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')}$$

Applies out of equilibrium too.

RTA (i)

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_A(\vec{p}) \quad S(\vec{p}', \vec{p}) = S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')}$$

Continue with our simplified collision integral. And with the eq. simplification for the transition rate.

$$\hat{C}f = \sum_{p'} S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')} f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_A(\vec{p})$$

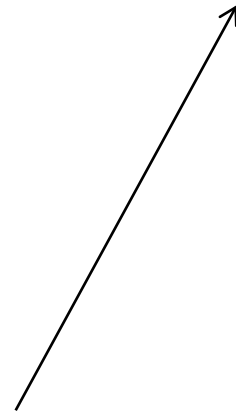
$$\hat{C}f(\vec{p}) = -f_A(\vec{p}) \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right) = -\frac{f_A(\vec{p})}{\tau_f}$$

$$\sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right) \equiv \frac{1}{\tau_f(\vec{p})}$$

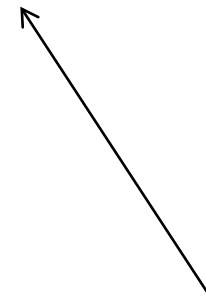
The characteristic time should be **independent** of f .

RTA (ii)

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$



even function of p'



odd function of p'

RTA (iii)

$$\frac{1}{\tau_f} = \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$

i) isotropic scattering: $S(\vec{p}, \vec{p}') = S(\vec{p}, -\vec{p}')$

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') = \frac{1}{\tau(\vec{p})}$$

The RTA is valid (assuming near-equilibrium conditions and MB statistics apply).

The characteristic time is the scattering time.

aside

$$\frac{1}{\tau(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \quad \text{scattering rate}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \frac{\Delta p_z}{p_z} = \sum_{p'} S(\vec{p}, \vec{p}') \frac{p_z - p'_z}{p_z} = \sum_{p'} S(\vec{p}, \vec{p}') \left[1 - \frac{p'_z}{p_z} \right]$$

$$\frac{1}{\tau_m(\vec{p})} = \frac{1}{\tau(\vec{p})}$$

isotropic scattering: RTA valid and $\tau_f = \tau = \tau_m$

RTA (iv)

$$\frac{1}{\tau_f} = \sum_{\vec{p}'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$

ii) **elastic scattering:** $E(\vec{p}') = E(\vec{p})$

$$\frac{1}{\tau_f} = \sum_{\vec{p}'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right) \quad (\text{for MB or FD statistics})$$

$$f_A = \tau_f(E) \left(-\frac{\partial f_0}{\partial E} \right) \vec{v} \cdot \vec{\mathcal{F}} \quad (\text{Lecture 15})$$

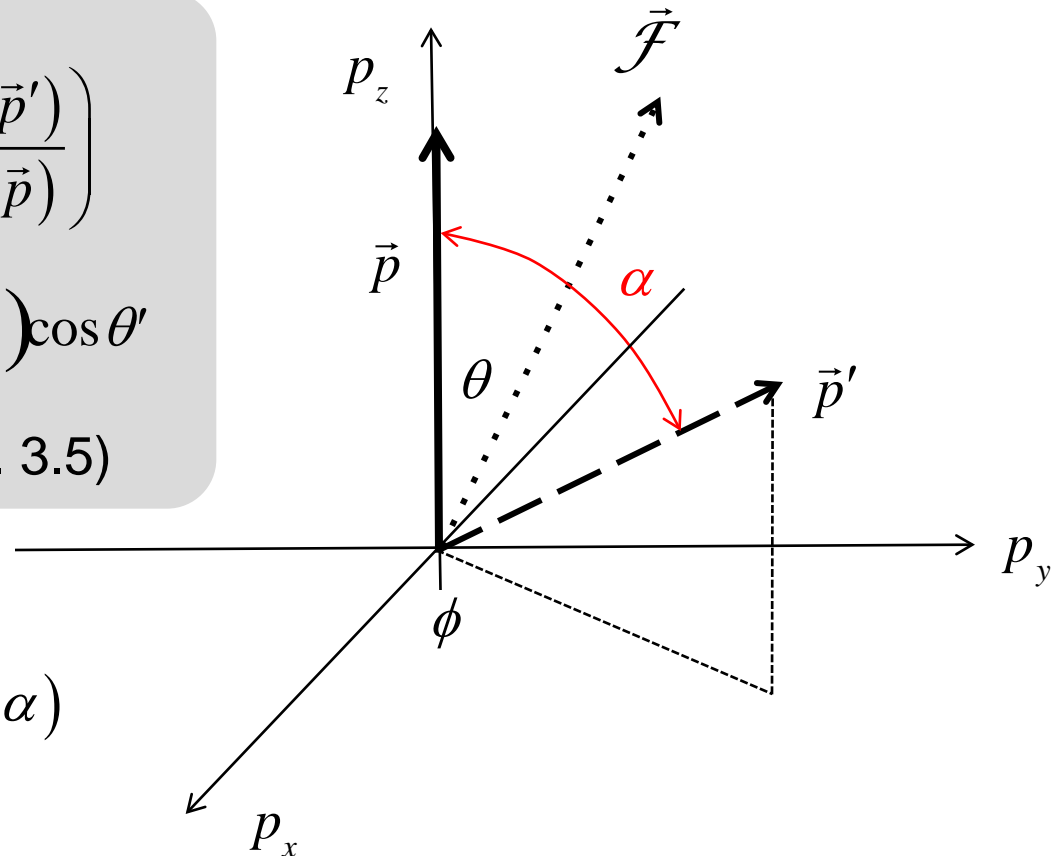
$$f_A = g(E) \cos \theta$$

RTA (v)

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$

$$f_A = g(E) \cos \theta \quad f'_A = g(E) \cos \theta'$$

(See Lundstrom, FCT Sec. 3.5)



$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') (1 - \cos \alpha)$$

The RTA is valid.

The characteristic time is ?

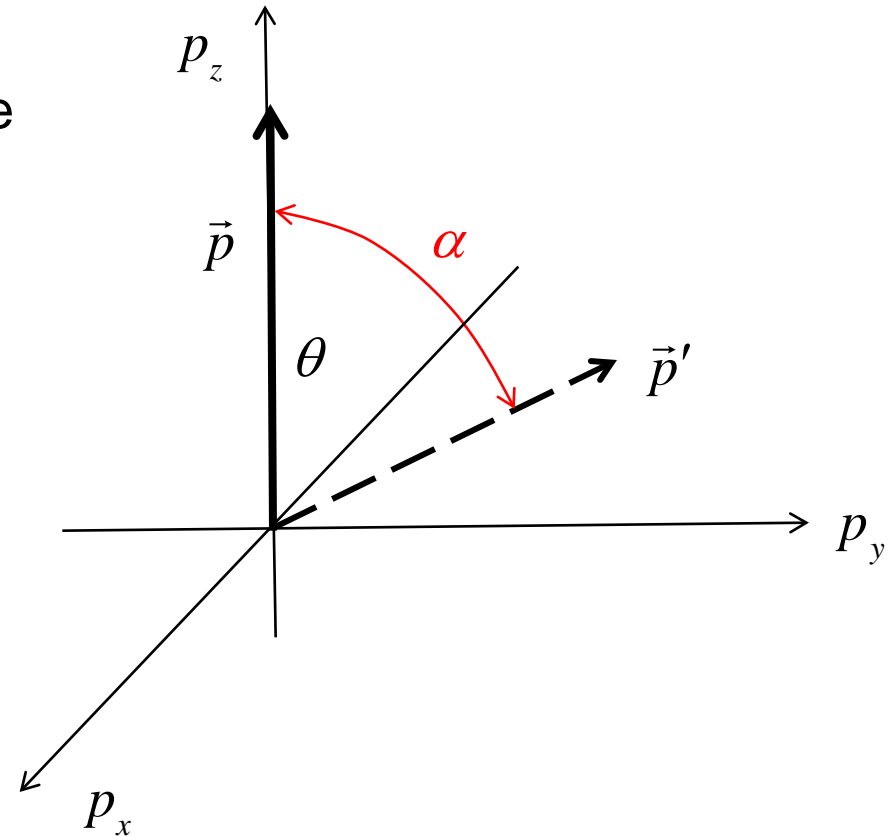
aside

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \frac{\Delta p_z}{p_z} \quad \text{momentum relaxation rate}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \frac{p_z - p'_z}{p_z}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \frac{p - p \cos \alpha}{p}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') (1 - \cos \alpha)$$



elastic scattering: RTA valid and

$$\tau_f = \tau_m$$

RTA (summary)

The (microscopic) momentum relaxation time is valid:

- 1) Near equilibrium
- 2) For isotropic scattering with MB statistics
- 3) For elastic scattering.

When valid, the characteristic time is the momentum relaxation time.

See Lundstrom, FCT□, Chapter 3, pp. 139-141 for more discussion.

exercise

Repeat the derivation for isotropic scattering but this time for Fermi-Dirac statistics. Does the RTA work for isotropic scattering and FD statistics?

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elastic scattering and FD statistics

Why does the RTA work for elastic scattering for either MS or FD statistics?

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') [1 - f(\vec{p})] - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p}) [1 - f(\vec{p}')]]$$

$$S(\vec{p}', \vec{p}) = S(\vec{p}, \vec{p}') \quad (\text{elastic scattering})$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}, \vec{p}') \{ f(\vec{p}') [1 - f(\vec{p})] - f(\vec{p}) [1 - f(\vec{p}')] \}$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}, \vec{p}') \{ f(\vec{p}') - f(\vec{p}) \} \quad (\text{same as for MB with elastic scattering})$$

Lundstrom, FCT prob. 3.7

Show that the RTA approximation describes the out-scattering processes, but not, in general, the in-scattering processes.

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p}) \quad (\text{Boltzmann statistics})$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - f(\vec{p}) \sum_{p'} S(\vec{p}, \vec{p}')$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \frac{f(\vec{p})}{\tau(\vec{p})}$$

Lundstrom, FCT prob. 3.7

Show that the RTA approximation describes the out-scattering processes, but not, in general, the in-scattering processes.

$$\hat{C}f(\vec{r}, \vec{p}, t) = -\frac{f(\vec{p}) - f_0(\vec{p})}{\tau(\vec{p})} \quad (\text{RTA})$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \frac{f_0(\vec{p})}{\tau(\vec{p})} - \frac{f(\vec{p})}{\tau(\vec{p})}$$

$$\hat{C}f(\vec{r}, \vec{p}, t) = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \frac{f(\vec{p})}{\tau(\vec{p})}$$

Lundstrom, FCT prob. 3.6

Does the RTA conserve carriers? $\hat{C}f(\vec{r}, \vec{p}, t) = -\frac{f_A(\vec{p})}{\tau_0}$

$$f_A(\vec{p}) = f(\vec{p}) - f_0(\vec{p})$$

$$\sum_{\vec{p}} \hat{C}f(\vec{r}, \vec{p}, t) = -\sum_{\vec{p}} \frac{f(\vec{p}) - f_0(\vec{p})}{\tau_0} = -\frac{(n - n_0)}{\tau_0}$$

$$f_A(\vec{p}) = f(\vec{p}) - f_s(\vec{p})$$

$$\sum_{\vec{p}} \hat{C}f(\vec{r}, \vec{p}, t) = -\sum_{\vec{p}} \frac{f(\vec{p}) - f_s(\vec{p})}{\tau_0} = -\frac{(n - n)}{\tau_0} = 0$$

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Lecture 17 HW

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{v} \times \vec{B} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = \left(-q\vec{\mathcal{E}} - q\vec{v} \times \vec{B} \right) \tau = m^* \vec{v}$$

$$\vec{v} = -\frac{q\tau}{m^*} \vec{\mathcal{E}} - \frac{q\tau}{m^*} \vec{v} \times \vec{B}$$

(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

$$v_x = -\frac{q\tau}{m^*} \mathcal{E}_x - \frac{q\tau}{m^*} v_y B_z$$

2D problem

z-directed B=field

$$v_y = -\frac{q\tau}{m^*} \mathcal{E}_y + \frac{q\tau}{m^*} v_x B_z$$

Lecture 17 HW: cont.

$$v_x = -\frac{q\tau}{m^*} \mathcal{E}_x - \frac{q\tau}{m^*} v_y B_z$$

$$v_y = -\frac{q\tau}{m^*} \mathcal{E}_y + \frac{q\tau}{m^*} v_x B_z$$

$$\omega_c = \frac{qB_z}{m^*}$$

“cyclotron frequency”

$$v_x = -\frac{q\tau}{m^*} \mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z - \left(\frac{q\tau}{m^*}\right)^2 v_x B_z^2$$

$$v_x \left(1 + \left(\frac{qB_z}{m^*}\right)^2 \tau^2\right) = -\frac{q\tau}{m^*} \mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z$$

$$v_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1 + (\omega_c \tau)^2}$$

Lecture 17 HW: cont.

$$v_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1 + (\omega_c \tau)^2}$$

$$J_x = -nqv_x = \frac{\sigma_n}{1 + (\mu_n B_z)^2} (\mathcal{E}_x - \mu_n B_z \mathcal{E}_y)$$

$$J_y = -nqv_y = \frac{\sigma_n}{1 + (\mu_n B_z)^2} (\mathcal{E}_x + \mu_n B_z \mathcal{E}_y)$$

$$\mu_n B_z = \frac{q\tau}{m^*} B_z = \omega_c \tau$$

magneto-conductivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\left(\omega_c \tau = \mu_n B_z \right)$$

$$J_i = \sigma_{ij} (B_z) \mathcal{E}_j$$

comments

$$\sigma_{ij}(B_z) = \frac{nq\mu_n}{1 + (\mu_n B_z)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}$$

1) A magnetic field affects both the diagonal and off-diagonal components of the magneto-conductivity tensor.

2) Small magnetic field means: $\mu_n B_z \ll 1$

$$\omega_c \tau \ll 1$$

magneto-resistivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \quad J_i = \sigma_{ij}(B_z) \mathcal{E}_j$$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{bmatrix} \sigma_L & -\sigma_T \\ \sigma_T & \sigma_L \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \begin{bmatrix} \rho_L & \rho_T \\ -\rho_T & \rho_L \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

$$\rho_L = \frac{\sigma_L}{\sigma_L^2 + \sigma_T^2} = \frac{1}{\sigma_n}$$
$$\rho_T = \frac{\sigma_T}{\sigma_L^2 + \sigma_T^2} = \frac{\mu_n B_z}{\sigma_n}$$

comparison

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \frac{1}{\sigma_n} \begin{bmatrix} 1 & \mu_n B_z \\ -\mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

- 1) Longitudinal magneto-resistance is independent of B
- 2) Hall voltage is proportional to B
- 3) BUT... we get the drift mobility, not the Hall mobility

questions

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