ECE-656: Fall 2011

Lecture 20:

Scattering II:
Relaxation time approximation

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outline

1. Justification of the RTA
2. Discussion
3. HW prob. 17

the BTE and the RTA

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = \hat{C} f$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = \sum_{p'} S\left(\vec{p}',\vec{p}\right) f\left(\vec{p}'\right)\left[1 - f\left(\vec{p}\right)\right] - \sum_{p'} S\left(\vec{p},\vec{p}'\right) f\left(\vec{p}\right)\left[1 - f\left(\vec{p}'\right)\right]$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = \sum_{p'} S\left(\vec{p}',\vec{p}\right) f\left(\vec{p}'\right) - \sum_{p'} S\left(\vec{p},\vec{p}'\right) f\left(\vec{p}\right)$$ (Boltzmann statistics)

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = -\frac{\left(f - f_S\right)}{\tau_f\left(\vec{r},\vec{p}\right)}$$

1) Under what conditions can we approximate the collision integral with the RTA?

2) When we can, how is τ_f defined?

symmetric and antisymmetric components

$$f\left(\vec{p}\right) = f_S\left(\vec{p}\right) + \delta f\left(\vec{p}\right)$$

$$f_S\left(\vec{p}\right) = f_S\left(-\vec{p}\right)$$ symmetric in momentum

$$\delta f\left(\vec{p}\right) = -\delta f\left(-\vec{p}\right)$$ anti-symmetric in momentum

$$f\left(\vec{p}\right) = f_S\left(\vec{p}\right) + f_A\left(\vec{p}\right)$$

simplified collision integral

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\, f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f(\vec{p})$$

(non-degenerate)

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\left[f_S(\vec{p}') + f_A(\vec{p}')\right] - \sum_{p'} S(\vec{p}, \vec{p}')\left[f_S(\vec{p}) + f_A(\vec{p})\right]$$

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\, f_S(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f_S(\vec{p}) = 0$$

(equilibrium)

$$\boxed{\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f_A(\vec{p})}$$

(assumes near-equilibrium – and MB statistics)

the “equilibrium simplification”

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p})$$

(return to the first form of the equation)

Can we relate $S(\vec{p}', \vec{p})$ and $S(\vec{p}, \vec{p}')$?

Yes, in equilibrium.

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_0(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_0(\vec{p}) = 0$$

(equilibrium)

$$S(\vec{p}', \vec{p}) f_0(\vec{p}') - S(\vec{p}, \vec{p}') f_0(\vec{p}) = 0$$

(detailed balance)

$$S(\vec{p}', \vec{p}) = S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')}$$

Applies out of equilibrium too.

RTA (i)

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_A(\vec{p}) \qquad S(\vec{p}', \vec{p}) = S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')}$$

Continue with our simplified collision integral.

And with the eq. simplification for the transition rate.

$$\hat{C}f = \sum_{p'} S(\vec{p}, \vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')} f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f_A(\vec{p})$$

$$\hat{C}f(\vec{p}) = -f_A(\vec{p}) \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right) = -\frac{f_A(\vec{p})}{\tau_f}$$

$$\sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right) \equiv \frac{1}{\tau_f(\vec{p})}$$

The characteristic time should be ***independent*** of *f*.

RTA (ii)

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}')} \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$

odd function of p’

even function of p’

RTA (iii)

$$\frac{1}{\tau_f} = \sum_{p'} S(\vec{p}, \vec{p}\,')\left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}\,')}\frac{f_A(\vec{p}\,')}{f_A(\vec{p})}\right)$$

i) isotropic scattering: $S(\vec{p}, \vec{p}\,') = S(\vec{p}, -\vec{p}\,')$

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') = \frac{1}{\tau(\vec{p})}$$

The RTA is valid (assuming near-equilibrium conditions and MB statistics apply).

The characteristic time is the scattering time.

aside

$$\frac{1}{\tau(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,')$$ scattering rate

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') \frac{\Delta p_z}{p_z} = \sum_{p'} S(\vec{p}, \vec{p}\,') \frac{p_z - p_z'}{p_z} = \sum_{p'} S(\vec{p}, \vec{p}\,') \left[1 - \frac{p_z'}{p_z}\right]$$

$$\frac{1}{\tau_m(\vec{p})} = \frac{1}{\tau(\vec{p})}$$

isotropic scattering: RTA valid and $\tau_f = \tau = \tau_m$

RTA (iv)

$$\frac{1}{\tau_f} = \sum_{p'} S\left(\vec{p}, \vec{p}'\right)\left(1 - \frac{f_0\left(\vec{p}\right)}{f_0\left(\vec{p}'\right)} \frac{f_A\left(\vec{p}'\right)}{f_A\left(\vec{p}\right)}\right)$$

ii) elastic scattering: $E\left(\vec{p}'\right) = E\left(\vec{p}\right)$

$$\frac{1}{\tau_f} = \sum_{p'} S\left(\vec{p}, \vec{p}'\right)\left(1 - \frac{f_A\left(\vec{p}'\right)}{f_A\left(\vec{p}\right)}\right)$$

(for MB **or** FD statistics)

$$f_A = \tau_f\left(E\right)\left(-\frac{\partial f_0}{\partial E}\right)\vec{\upsilon} \bullet \vec{\mathcal{F}}$$

(Lecture 15)

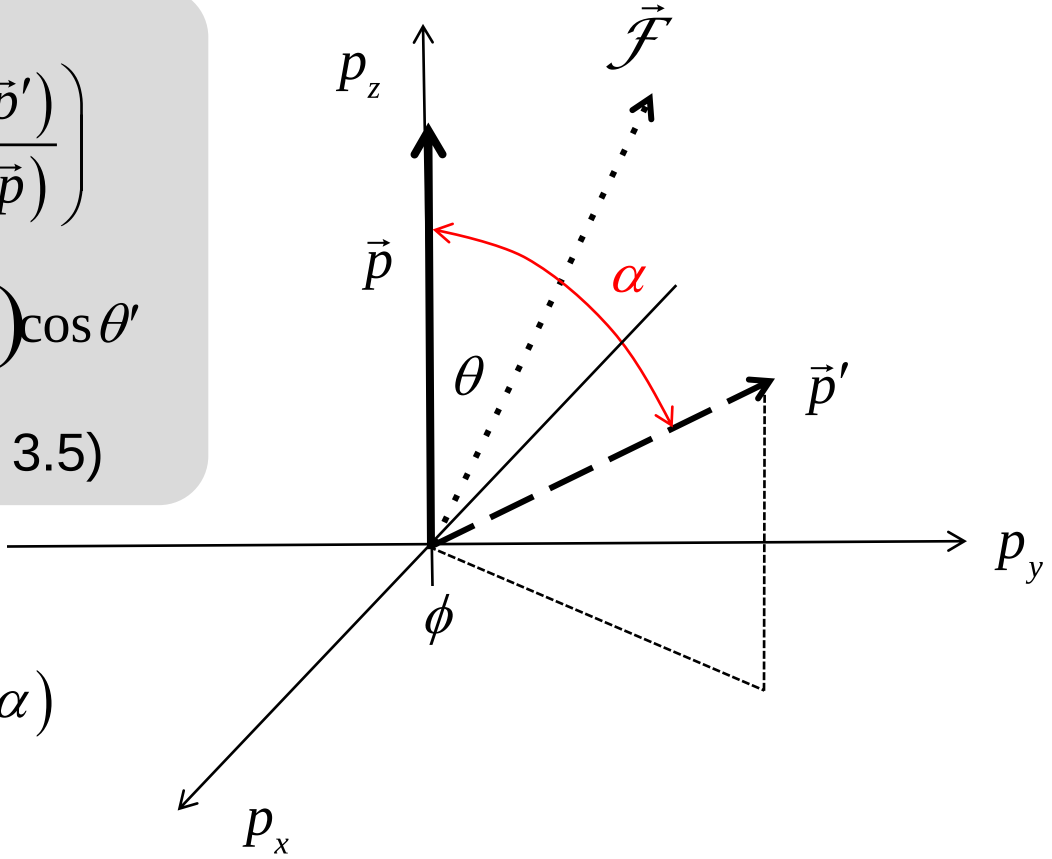
$$f_A = g\left(E\right)\cos\theta$$

RTA (v)

$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}') \left(1 - \frac{f_A(\vec{p}')}{f_A(\vec{p})} \right)$$

$$f_A = g(E)\cos\theta \qquad f_A' = g(E)\cos\theta'$$

(See Lundstrom, FCT Sec. 3.5)



$$\frac{1}{\tau_f(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}')(1 - \cos\alpha)$$

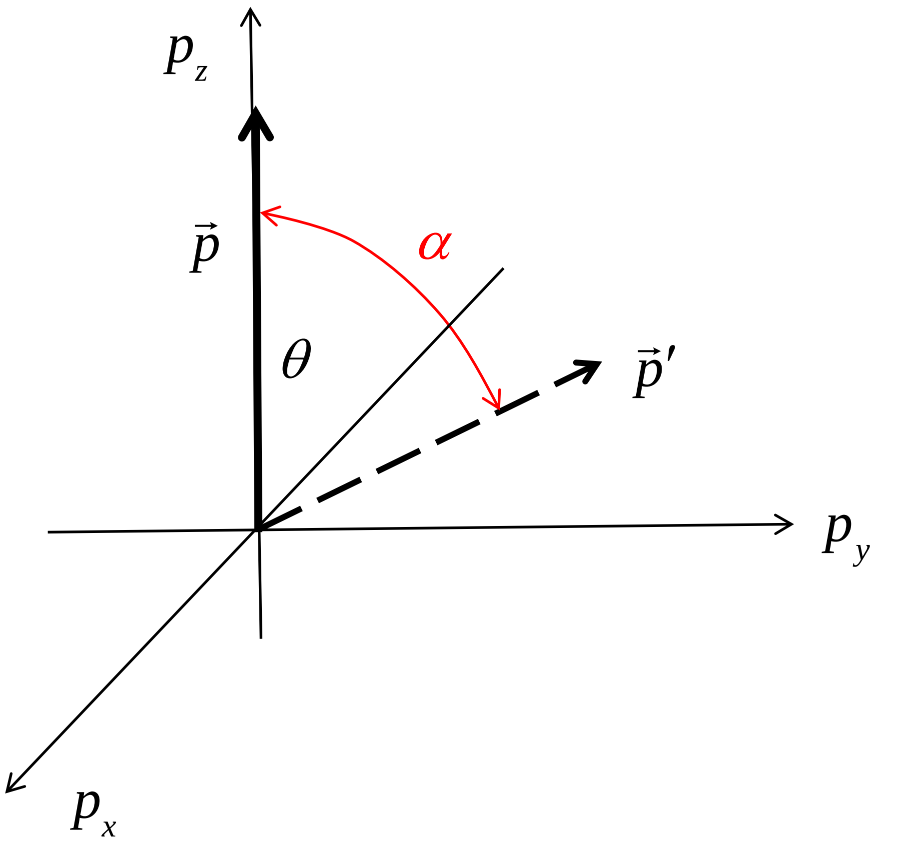
The RTA is valid.

The characteristic time is ?

aside

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') \frac{\Delta p_z}{p_z}$$

momentum relaxation rate



$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') \frac{p_z - p_z'}{p_z}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') \frac{p - p\cos\alpha}{p}$$

$$\frac{1}{\tau_m(\vec{p})} = \sum_{p'} S(\vec{p}, \vec{p}\,') (1 - \cos\alpha)$$

elastic scattering: RTA valid and $\tau_f = \tau_m$

RTA (summary)

The (microscopic) momentum relaxation time is valid:

1) Near equilibrium

2) For isotropic scattering with MB statistics

3) For elastic scattering.

When valid, the characteristic time is the momentum relaxation time.

See Lundstrom, FCT□, Chapter 3, pp. 139-141 for more discussion.

exercise

Repeat the derivation for isotropic scattering but this time for Fermi-Dirac statistics. Does the RTA work for isotropic scattering and FD statistics?

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elastic scattering and FD statistics

Why does the RTA work for elastic scattering for either MS or FD statistics?

$$\hat{C}f(\vec{r},\vec{p},t) = \sum_{p'} S(\vec{p}',\vec{p})\, f(\vec{p}')\left[1 - f(\vec{p})\right] - \sum_{p'} S(\vec{p},\vec{p}')\, f(\vec{p})\left[1 - f(\vec{p}')\right]$$

$$S(\vec{p}',\vec{p}) = S(\vec{p},\vec{p}')$$ (elastic scattering)

$$\hat{C}f(\vec{r},\vec{p},t) = \sum_{p'} S(\vec{p},\vec{p}')\left\{ f(\vec{p}')\left[1 - f(\vec{p})\right] - f(\vec{p})\left[1 - f(\vec{p}')\right]\right\}$$

$$\hat{C}f(\vec{r},\vec{p},t) = \sum_{p'} S(\vec{p},\vec{p}')\left\{ f(\vec{p}') - f(\vec{p})\right\}$$ (same as for MB with elastic scattering)

Lundstrom, FCT prob. 3.7

Show that the RTA approximation describes the out-scattering processes, but not, in general, the in-scattering processes.

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}',\vec{p}\right)f\left(\vec{p}'\right)-\sum_{p'}S\left(\vec{p},\vec{p}'\right)f\left(\vec{p}\right)$$ (Boltzmann statistics)

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}',\vec{p}\right)f\left(\vec{p}'\right)-f\left(\vec{p}\right)\sum_{p'}S\left(\vec{p},\vec{p}'\right)$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right)=\sum_{p'}S\left(\vec{p}',\vec{p}\right)f\left(\vec{p}'\right)-\frac{f\left(\vec{p}\right)}{\tau\left(\vec{p}\right)}$$

Lundstrom, FCT prob. 3.7

Show that the RTA approximation describes the out-scattering processes, but not, in general, the in-scattering processes.

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = -\frac{f\left(\vec{p}\right) - f_0\left(\vec{p}\right)}{\tau\left(\vec{p}\right)} \qquad \text{(RTA)}$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = \frac{f_0\left(\vec{p}\right)}{\tau\left(\vec{p}\right)} - \frac{f\left(\vec{p}\right)}{\tau\left(\vec{p}\right)}$$

$$\hat{C}f\left(\vec{r},\vec{p},t\right) = \sum_{p'} S\left(\vec{p}',\vec{p}\right) f\left(\vec{p}'\right) - \frac{f\left(\vec{p}\right)}{\tau\left(\vec{p}\right)}$$

Lundstrom, FCT prob. 3.6

Does the RTA conserve carriers? $\hat{C}f\left(\vec{r},\vec{p},t\right) = -\dfrac{f_A\left(\vec{p}\right)}{\tau_0}$

$$f_A\left(\vec{p}\right) = f\left(\vec{p}\right) - f_0\left(\vec{p}\right)$$

$$\sum_{\vec{p}} \hat{C}f\left(\vec{r},\vec{p},t\right) = -\sum_{\vec{p}} \frac{f\left(\vec{p}\right) - f_0\left(\vec{p}\right)}{\tau_0} = -\frac{\left(n - n_0\right)}{\tau_0}$$

$$f_A\left(\vec{p}\right) = f\left(\vec{p}\right) - f_S\left(\vec{p}\right)$$

$$\sum_{\vec{p}} \hat{C}f\left(\vec{r},\vec{p},t\right) = -\sum_{\vec{p}} \frac{f\left(\vec{p}\right) - f_S\left(\vec{p}\right)}{\tau_0} = -\frac{\left(n - n\right)}{\tau_0} = 0$$

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3. **HW prob. 17**

Lecture 17 HW

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = \left(-q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}\right)\tau = m^*\vec{\upsilon}$$

$$\vec{\upsilon} = -\frac{q\tau}{m^*}\vec{\mathcal{E}} - \frac{q\tau}{m^*}\vec{\upsilon} \times \vec{B}$$

(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x - \frac{q\tau}{m^*}\upsilon_y B_z$$

$$\upsilon_y = -\frac{q\tau}{m^*}\mathcal{E}_y + \frac{q\tau}{m^*}\upsilon_x B_z$$

2D problem

z-directed B=field

Lecture 17 HW: cont.

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x - \frac{q\tau}{m^*}\upsilon_y B_z$$

$$\upsilon_y = -\frac{q\tau}{m^*}\mathcal{E}_y + \frac{q\tau}{m^*}\upsilon_x B_z$$

$$\omega_c = \frac{qB_z}{m^*}$$

"cyclotron frequency"

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z - \left(\frac{q\tau}{m^*}\right)^2 \upsilon_x B_z^2$$

$$\upsilon_x\left(1+\left(\frac{qB_z}{m^*}\right)^2\tau^2\right) = -\frac{q\tau}{m^*}\mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z$$

$$\upsilon_x = \frac{-\mu_n\mathcal{E}_x + \mu_n^2\mathcal{E}_y B_z}{1+(\omega_c\tau)^2}$$

Lecture 17 HW: cont.

$$\upsilon_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1+(\omega_c \tau)^2}$$

$$J_x = -nq\upsilon_x = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_x - \mu_n B_z \mathcal{E}_y\right)$$

$$J_y = -nq\upsilon_y = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_x + \mu_n B_z \mathcal{E}_y\right)$$

$$\mu_n B_z = \frac{q\tau}{m^*} B_z = \omega_c \tau$$

magneto-conductivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\left(\omega_c \tau = \mu_n B_z\right)$$

$$J_i = \sigma_{ij}\left(B_z\right)\mathcal{E}_j$$

comments

$$\sigma_{ij}\left(B_z\right)=\frac{nq\mu_n}{1+\left(\mu_n B_z\right)^2}\begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}$$

1) A magnetic field affects both the diagonal and off-diagonal components of the magneto-conductivity tensor.

2) Small magnetic field means: $\mu_n B_z << 1$

$$\omega_c \tau << 1$$

magneto-resistivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$J_i = \sigma_{ij}\left(B_z\right)\mathcal{E}_j$$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{bmatrix} \sigma_L & -\sigma_T \\ \sigma_T & \sigma_L \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \begin{bmatrix} \rho_L & \rho_T \\ -\rho_T & \rho_L \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

$$\rho_L = \frac{\sigma_L}{\sigma_L^2 + \sigma_T^2} = \frac{1}{\sigma_n}$$

$$\rho_T = \frac{\sigma_T}{\sigma_L^2 + \sigma_T^2} = \frac{\mu_n B_z}{\sigma_n}$$

comparison

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \frac{1}{\sigma_n} \begin{bmatrix} 1 & \mu_n B_z \\ -\mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

1) Longitudinal magneto-resistance is independent of *B*
2) Hall voltage is proportional to *B*
3) BUT… we get the drift mobility, not the Hall mobility

questions

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