KRAMERS-KRÖNIG

(INDEPENDENT RESEARCH)

KEY ASSUMPTION (ISOTROPIC SYSTEMS)

$$4\pi P(t) = \int_0^\infty F(t')\, E(t-t')\, dt'$$

IMPORTANT POINTS:

1. LINEAR RELATIONSHIP BETWEEN POLARIZATION & ELECTRIC FIELD

2. CAUSALITY
   P @ t DEPENDS ON VALUES OF E @ EARLIER TIMES

3. 4π CONVENIENCE (UNITS)

4. LOCAL RELATIONSHIP
   $P(\vec{r})$ DEPENDS ON $\vec{E}(\vec{r})$

IF QUANTITIES DEPEND ON TIME LIKE $e^{-i\omega t}$

$$\Rightarrow \quad 4\pi P = E \int_0^\infty F(t')\, e^{i\omega t'}\, dt'$$

OR, SINCE $D = E + 4\pi P$

$$\boxed{\epsilon = 1 + \int_0^\infty F(t')\, e^{i\omega t'}\, dt'} \qquad (*)$$

TAKE $\quad \epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$

$$\Rightarrow \quad \epsilon(-\omega) = \epsilon^*(\omega)$$

FIELDS ARE REAL

$$\begin{cases} \epsilon_1^*(-\omega) = \epsilon_1^*(\omega) & \text{REAL PART IS } \underline{\text{EVEN}} \\ \epsilon_2(-\omega) = -\epsilon_2(\omega) & \text{IMAG. PART IS } \underline{\text{ODD}} \end{cases}$$

FROM $(*)$

$$\epsilon_1(\omega) = 1 + \int_0^\infty F(t') \cos\omega t'\, dt'$$

$$\epsilon_2(\omega) = \int_0^\infty F(t') \sin\omega t'\, dt'$$

LET US NOW IMPOSE THE CONDITION

$$F(t') = 0 \qquad t' < 0$$

(THIS DOESN'T CHANGE ANYTHING) EXCEPT THAT WE CAN CHANGE THE LIMITS OF INTEGRATION

THEN, IF WE INVERT THE FOURIER TRANSFORM, WE GET THAT

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\epsilon(\omega) - 1]\, e^{-i\omega t}\, d\omega$$

$$\equiv \frac{1}{\pi} \int_0^{\infty} [\underbrace{(\epsilon_1 - 1)}_{\text{EVEN}} \cos\omega t + \underbrace{\epsilon_2}_{\text{ODD}} \sin\omega t]\, d\omega$$

IF, FOR $t > 0$

$$\int_0^{\infty} (\epsilon_1 - 1) \cos\omega t\, d\omega \equiv \int_0^{\infty} \epsilon_2 \sin\omega t\, d\omega$$

$$\Rightarrow \quad F \equiv 0 \qquad t < 0$$

FROM THERE, ONE CAN SHOW (HOMEWORK)

$$\begin{cases} \epsilon_1 = 1 + \dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{\epsilon_2(\omega')}{\omega' - \omega}\, d\omega' \\ \epsilon_2 = -\dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{\epsilon_1(\omega') - 1}{\omega' - \omega}\, d\omega' \end{cases}$$

POLE @ $\omega = 0$

$\frac{4\pi\sigma}{\omega}$

KK

OTHER EQUIVALENT EXPRESSIONS

$$\begin{cases} \epsilon_1 = 1 + \dfrac{2}{\pi} \displaystyle\int_0^{\infty} d\omega' \; \epsilon_2(\omega') \left[\dfrac{\omega'}{\omega'^2 - \omega^2} \right] \\[2ex] \epsilon_2 = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} d\omega' \; \epsilon_1(\omega') \left\{ \dfrac{\omega}{\omega^2 - \omega'^2} \right\} \end{cases}$$

ALSO EXPRESSIONS RELATING n TO K

MORE ELEGANT DERIVATION

SINCE

$$\epsilon = 1 + \int_0^{\infty} F(t')\, e^{i\omega t'} \, dt'$$

PROVIDED $F(t') \to 0$ FOR $t' \to \infty$

ϵ IS ANALYTICAL IN UPPER HALF OF COMPLEX PLANE

THEN, USING CAUCHY'S THEOREM (COMPLEX VARIABLES)

$$\epsilon(\omega) = -\frac{i}{2\pi} \oint_C \frac{\epsilon(\omega') - 1}{\omega' - \omega} \, d\omega'$$

SEE JACKSON P. 332

SOME GRAL. PPTIES OF THE PERMITTIVITY (DIELECTRIC CONSTANT)

START WITH

$$\epsilon_1 = 1 + \frac{2}{\pi} \int_0^\infty \epsilon_2(\omega') \frac{\omega'}{\omega'^2 - \omega^2} d\omega'$$

FOR $\hbar\omega \gg$ energy of interband transitions

EXPLAIN

$$\epsilon_1(\omega) \approx 1 - \frac{2}{\pi\omega^2} \int_0^\infty \epsilon_2(\omega')\,\omega'\, d\omega'$$

$$\equiv 1 - \frac{8}{\omega^2} \int_0^\infty \sigma_1(\omega')\, d\omega'$$

$$\Rightarrow \boxed{\epsilon_1(\omega) \equiv 1 - \frac{\omega_p^2}{\omega^2}}$$

VALID FOR ALL SOLIDS !

$$\omega_p^2 = 8 \int_0^\infty \sigma_1(\omega')\, d\omega'$$

WE WILL LATER SEE THAT

$$\omega_p^2 = \frac{4\pi N e^2}{m} \qquad \omega_p \approx 20\,\text{eV}$$

DRUDE MODEL (METALS)

ASSUMPTION : DELTA PULSE OF ELECTRIC FIELD LEADS TO CURRENT DENSITY OF THE FORM

$$\boxed{J = J_0\, e^{-\gamma t}}$$

SINCE $J(t) = \dfrac{\partial P}{\partial t} = \dfrac{1}{4\pi}\dfrac{\partial}{\partial t}\displaystyle\int_0^{\infty} F(t')\, E(t-t')\, dt'$

$$= \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\iint_0^{\infty} F(t')\, e^{-i\Omega(t-t')} \underbrace{E_\Omega}_{\text{INDEPENDENT OF } \Omega \text{, IF } E \propto \delta(t)} d\Omega\, dt'\right]$$

$$\boxed{E(t) = 2\pi E_\Omega\, \delta(t)}$$

$$= \frac{-i\Omega}{4\pi} E_\Omega \int_{-\infty}^{\infty} e^{-i\Omega t}\, d\Omega \underbrace{\int_0^{\infty} F(t')\, e^{i\Omega t'}\, dt'}_{[\epsilon(\Omega)-1]}$$

$$\Rightarrow J(t) = E_\Omega \int_{-\infty}^{\infty} e^{-i\Omega t}\, \sigma(\Omega)\, d\Omega$$

$\times e^{i\Omega' t} dt$ + INTEGRATE USING THAT $\displaystyle\int_{-\infty}^{\infty} e^{-i\alpha\beta}\, d\beta = 2\pi\delta$

$$\sigma(\Omega') = \frac{1}{2\pi E_\Omega}\int_0^{\infty} J(t)\, e^{i\Omega' t}\, dt$$

$$\sigma(\omega) \propto \frac{1}{\gamma - i\omega}$$

THE PROPORTIONALITY CONSTANT CAN BE ESTIMATED USING THAT

$$\sigma(\omega=0) \equiv \frac{Ne^2}{m} \qquad \gamma \equiv \frac{1}{\tau}$$

RELAXATION TIME

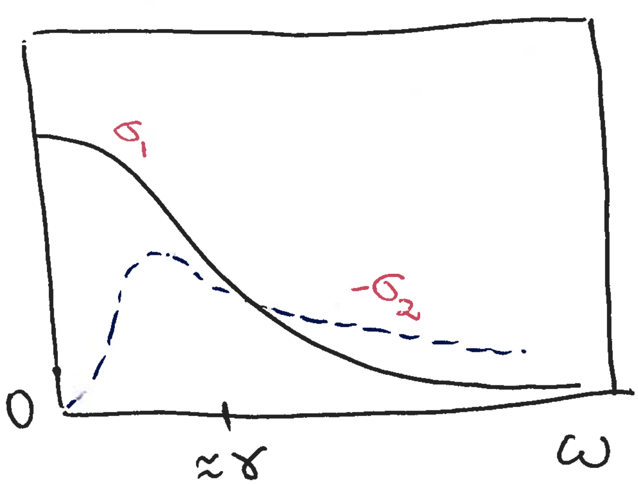
WE ~~THEN~~ HAVE

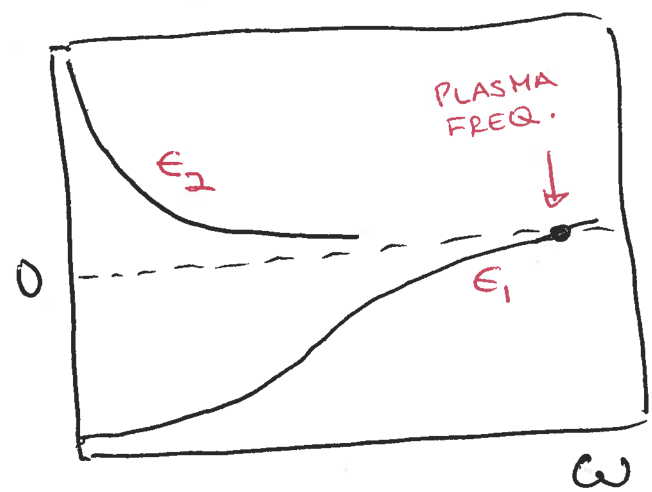
$$\epsilon_1(\omega) = 1 + \frac{4\pi}{\omega}\sigma_2(\omega) = 1 - \frac{4\pi Ne^2}{m}\frac{1}{[\omega^2+\gamma^2]}$$

CORRECT BEHAVIOR @ $\omega \to \infty$

$$\epsilon_2(\omega) = \frac{4\pi Ne^2}{m}\frac{\gamma}{\omega[\omega^2+\gamma^2]}$$

DIVERGES FOR $\omega \to 0$
VERY IMPORTANT





LORENTZ MODEL

(CLASSICAL OSCILLATORS OR ASSEMBLY OF INDEPENDENT NEUTRAL ATOMS)

LATER QM

... PHONONS, EXCITONS

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{-eE_0}{m} e^{-i\omega t}$$

$$\Rightarrow \quad x = \frac{-eE_0}{m} e^{-i\omega t} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

THE ELECTRIC-DIPOLE MOMENT PER UNIT OF VOLUME, i.e., THE POLARIZATION IS:

$$P_x = -Nex$$

SINCE $\quad \epsilon(\omega) = 1 + 4\pi P_x / E_x$

WE GET

$$\boxed{\epsilon(\omega) = 1 + \frac{4\pi N e^2}{m} \left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1}}$$

LORENTZIAN

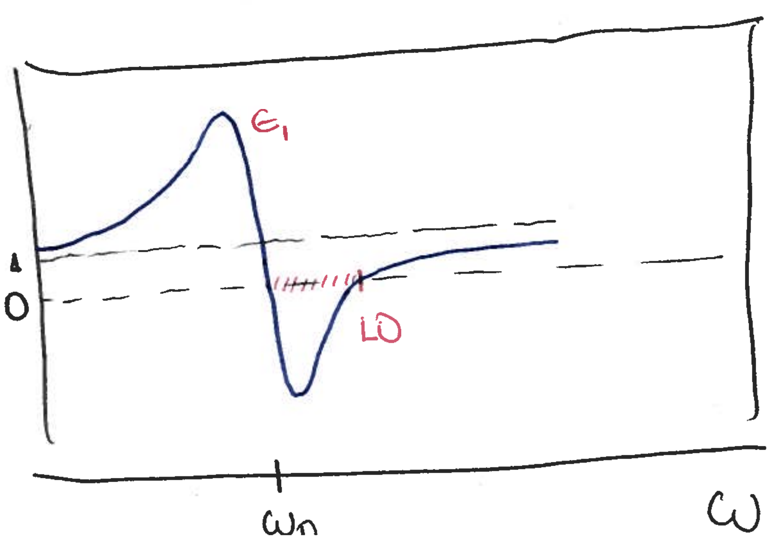
$$\begin{cases} \epsilon_1(\omega) = 1 + \dfrac{\omega_p^2\,(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \\[2em] \epsilon_2(\omega) = \dfrac{\omega_p^2\,\omega\,\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \approx \dfrac{\omega_p^2\,(\gamma/4\omega_0)}{(\omega-\omega_0)^2+\gamma^2/4} \end{cases}$$

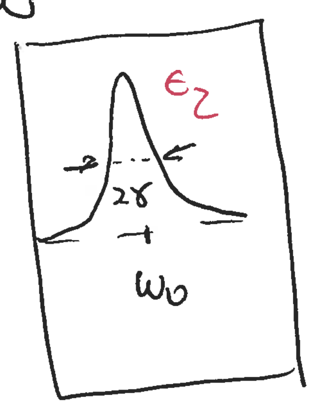
VERY DIFFERENT FROM METALS !
AT LOW FREQS.

$$\omega \to 0 \qquad \begin{cases} \epsilon_1 \to 1 + \dfrac{\omega_p^2}{\omega_0^2} > 1 \\ \epsilon_2 \to 0 \end{cases}$$

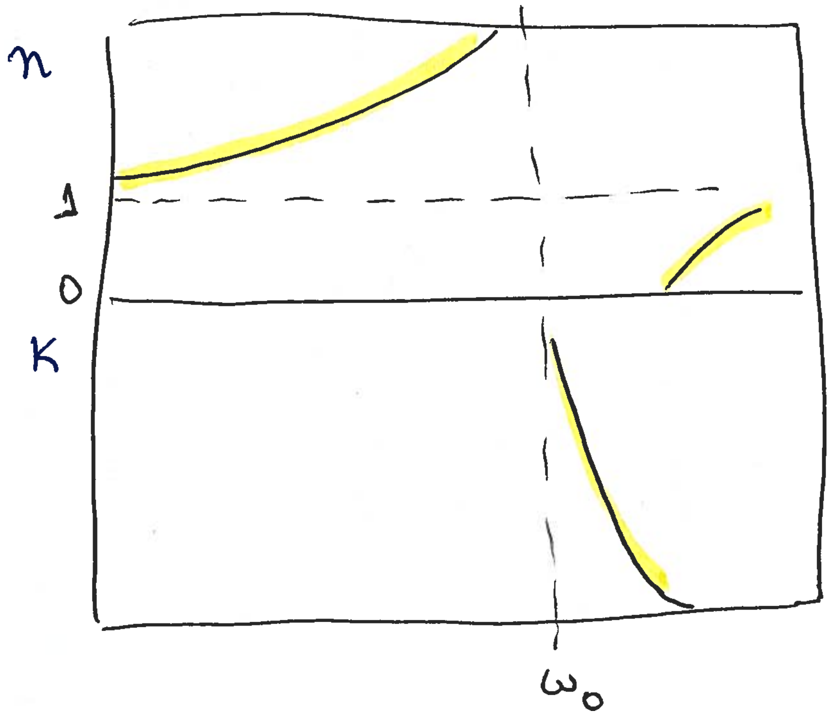
$$\omega \to \infty \qquad \begin{cases} \epsilon_1 \to 1 - \dfrac{\omega_p^2}{\omega^2} \\ \epsilon_2 \to \dfrac{\omega_p^2\,\gamma}{\omega^3} \end{cases}$$

IDENTICAL TO METALS





HOW ABOUT THE REFRACTIVE INDEX?



REFLECTIVITY?