Review of EM Theory

(1) Maxwell's Equations

$$\vec{e}(\vec{r},t) = \mathrm{Re}[\vec{E}(\vec{r})e^{j\omega t}] \qquad \vec{h}(\vec{r},t) = \mathrm{Re}[\vec{H}(\vec{r})e^{j\omega t}]$$

$$\vec{j}(\vec{r},t) = \mathrm{Re}[\vec{J}(\vec{r})e^{j\omega t}] \qquad \vec{p}(\vec{r},t) = \mathrm{Re}[\vec{P}(\vec{r})e^{j\omega t}]$$

① $\vec{\nabla}\cdot(\varepsilon_0\vec{e} + \vec{p}) = \rho$ - Gauss' Law

$$\vec{d} = \varepsilon_0\vec{e} + \vec{p}$$

$$\dot{\nabla}\cdot d = \rho$$

$\vec{d}$ is displacement vector

$\vec{m}$, $\vec{b}$ are magnetization & magnetic induction

$\varepsilon_0 = 8.854\times10^{-12}\ F/m$ is permittivity of free space

$\mu_0 = 4\pi\times10^{-7}\ H/m$ is permeability of free space

② $\vec{\nabla}\cdot\vec{b} = 0 \qquad \vec{b} = \mu_0(\vec{h} + \vec{m})$

$$\vec{\nabla}\cdot\vec{A} = (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(3) $$\vec{\nabla} \times \vec{h} = \vec{j} + \varepsilon_0 \frac{\partial \vec{e}}{\partial t} + \frac{\partial \vec{p}}{\partial t} = \vec{j} + \frac{\partial \vec{d}}{\partial t}$$ - Ampere's Law

(4) $$\vec{\nabla} \times \vec{e} = -\mu_0 \frac{\partial \vec{h}}{\partial t} - \mu_0 \frac{\partial \vec{m}}{\partial t} = -\frac{\partial \vec{b}}{\partial t}$$ - Faraday's Law

Phasor Representation

$$\frac{\partial}{\partial t} \Rightarrow j\omega$$

$$\vec{d}(\vec{r},t) = \mathrm{Re}[\vec{D}(\vec{r})e^{j\omega t}]$$
$$\vec{b}(\vec{r},t) = \mathrm{Re}[\vec{B}(\vec{r})e^{j\omega t}]$$

$$\vec{\nabla} \times \vec{\mathrm{H}} = \vec{J} + j\omega\varepsilon_0\vec{E} + j\omega\vec{P}$$
$$\vec{\nabla} \times \vec{E} = -j\omega\mu_0\vec{H}$$ [where $\vec{m} = 0$ (no magnetization)]

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$ [constitutive equation]

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\vec{P}(\omega) = \varepsilon_0 \chi(\omega) \vec{E}(\omega)$$

$$\vec{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 n^2 \vec{E} = \varepsilon \vec{E}$$

$\varepsilon_r = 1 + \chi = n^2$ (Relative Dielectric Constant)

$$\varepsilon \equiv \varepsilon_0 \varepsilon_r$$

(2) Wave Equation

Free Space $\vec{j} = 0, \vec{p} = 0$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{e}) = \vec{\nabla} \times (-\mu_0 \frac{\partial \vec{h}}{\partial t}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{h}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{e}}{\partial t^2}$$

using: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ and $\vec{\nabla} \cdot \vec{e} = 0$

then:

$$\nabla^2 \vec{e} - \frac{1}{c^2} \frac{\partial^2 \vec{e}}{\partial t^2} = 0$$

where $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

similarly:

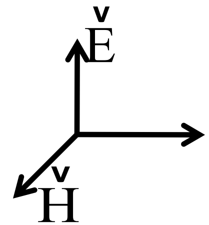
$$\nabla^2 \vec{h} - \frac{1}{c^2} \frac{\partial^2 \vec{h}}{\partial t^2} = 0$$

$$\nabla^2 \vec{f} = \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2}$$

$$\vec{\nabla} \times \vec{\mathrm{H}} = j\omega\varepsilon_0 \vec{\mathrm{E}} \rightarrow \vec{k}_0 \times \vec{\mathrm{H}} = -\omega\varepsilon_0 \vec{\mathrm{E}} \qquad (\vec{\mathrm{E}} \perp \vec{k}_0 \text{ and } \vec{\mathrm{H}})$$

$$\vec{\nabla} \times \vec{\mathrm{E}} = -j\omega\mu_0 \vec{\mathrm{H}} \rightarrow \vec{k}_0 \times \vec{\mathrm{E}} = \omega\mu_0 \vec{\mathrm{H}} \qquad \left(\vec{\mathrm{H}} \perp \vec{k}_0 \text{ and } \vec{\mathrm{E}}\right)$$

$\Rightarrow$



$$\vec{k}_0 = \frac{\omega}{c}\hat{k}$$

$$\left|\frac{\vec{\mathrm{E}}}{\vec{\mathrm{H}}}\right| = \frac{\mathrm{E}}{\mathrm{H}} = \frac{\omega\mu_0}{k_0} = \frac{\omega\mu_0}{\omega}c = \frac{\mu_0}{\left(\mu_0\varepsilon_0\right)^{1/2}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 = 377\Omega$$

Wave Impedance in free space

Poynting Vector:

$$\boxed{\vec{S} = \frac{1}{2}\vec{\mathrm{E}} \times \vec{\mathrm{H}}^*} \qquad \left(\vec{S} \parallel \vec{k}_0\right)$$

$$\vec{S} = \frac{1}{2}\vec{\mathrm{E}} \times \vec{\mathrm{H}}^* = \frac{1}{2}\vec{\mathrm{E}} \times \frac{\left(\vec{k}_0 \times \vec{\mathrm{E}}^*\right)}{\omega\mu_0} = \frac{1}{2}\left|\vec{\mathrm{E}}\right|^2 \frac{\vec{k}_0}{\omega\mu_0}$$

using $\vec{\mathrm{A}} \times \left(\vec{\mathrm{B}} \times \vec{C}\right) = \vec{\mathrm{B}}\left(\vec{\mathrm{A}} \cdot \vec{C}\right) - \vec{C}\left(\vec{\mathrm{A}} \cdot \vec{\mathrm{B}}\right)$

(3) Waves in dielectrics

e.g. Laser Medium: Ruby Laser (Active Medium – Chromium (5%) in a dielectric host of Al_2O_3

$$\vec{\nabla} \times \vec{h} = \varepsilon_0 \frac{\partial \vec{e}}{\partial t} + \frac{\partial \vec{p}_l}{\partial t} + \frac{\partial \vec{p}_a}{\partial t} = \varepsilon_0 n^2 \frac{\partial \vec{e}}{\partial t} + \frac{\partial \vec{p}_a}{\partial t}$$

$$n^2 = \left(1 + \chi_l\right)$$

$$\vec{\nabla} \times \vec{e} = -\mu_0 \frac{\partial \vec{h}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{e} = -\mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 n^2 \frac{\partial \vec{e}}{\partial t} + \frac{\partial \vec{p}_a}{\partial t}\right)$$

using $\vec{\nabla}\left(\vec{\nabla} \cdot \vec{e}\right) - \nabla^2 \vec{e}$ $\left(\vec{\nabla} \cdot \vec{d} = \varepsilon_0 \varepsilon_r \vec{\nabla} \cdot \vec{e} = 0\right)$

$$\Longrightarrow \boxed{\nabla^2 \vec{e} - \frac{n^2}{c^2} \frac{\partial^2 \vec{e}}{\partial t} = \mu_0 \frac{\partial^2 \vec{p}_a}{\partial t^2}}$$

← Source for the Optical Fields

Propagation velocity $v = \frac{c}{n}$

$\vec{k}_0 \longrightarrow \vec{k}$ $k = n\frac{\omega}{c} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \sqrt{\varepsilon_0 \mu_0}\,\omega = \sqrt{\varepsilon \mu_0}\,\omega = \sqrt{\varepsilon_r} \sqrt{\varepsilon_0 \mu_0}\,\omega$