Propagation of Gaussian Beams

$$\nabla^2 e - \left(\frac{n}{c}\right)\frac{\partial^2 e}{\partial t^2} = 0 \qquad \vec{e}\left(\vec{r},t\right) = E\left(x,y,z\right)e^{j\omega t}$$

$$\nabla^2 E + \frac{\omega^2}{c^2} n^2 E = 0 \qquad k = \frac{n\omega}{c}$$

$$\nabla^2 = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}_{\nabla_t^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 E + k^2 E = 0 \qquad E\left(x,y,z\right) = E_0 \Psi\left(x,y,z\right)e^{-jkz}$$

$$\frac{\partial E}{\partial z} = E_0\left[\frac{\partial \Psi}{\partial z}e^{jkz} - jk\Psi e^{-jkz}\right]$$

$$\frac{\partial^2 E}{\partial z^2} = E_0\left[\frac{\partial^2 \Psi}{\partial z^2}e^{-jkz} - 2jk\frac{\partial \Psi}{\partial z}e^{-jkz} + \left(-jk\right)^2 \Psi e^{-jkz}\right]$$

cancels $k^2 E$

$$\frac{\partial^2 \Psi}{\partial z^2} << \frac{1}{\lambda}\frac{\partial \Psi}{\partial z}$$

$$\nabla_t^2 \Psi - 2jk\frac{\partial \Psi}{\partial z} = 0 \qquad \nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$TEM_{0,0}$ mode (cylindrically symmetric)

$$\Psi_0 = \exp\left(-j\left[P\left(z\right) + \frac{kr^2}{2q(z)}\right]\right)$$

Gaussian profile
curvature of phase front

Complex phase shift

Complex beam parameter

$$\Psi_0 = \exp\left(-j\left[P(z)+\frac{kr^2}{2q(z)}\right]\right) \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\Psi - 2jk\frac{\partial\Psi}{\partial z} = 0$$

$$-2jk\frac{\partial\Psi_0}{\partial z} = \left[-2kP'(z)+\frac{k^2r^2}{q^2(z)}q'(z)\right]\Psi_0$$

$$\frac{\partial\Psi_0}{\partial r} = -j\frac{kr}{q(z)}\Psi_0 \qquad r\frac{\partial\Psi_0}{\partial r} = -j\frac{kr^2}{q(z)}\Psi_0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial\Psi_0}{\partial r}\right) = -j\frac{kr^2}{q(z)}\left[-j\frac{kr}{q(z)}\right]\Psi_0 - j2\frac{kr}{q(z)}\Psi_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi_0}{\partial r}\right) = \left[-\frac{k^2r^2}{q^2(z)} - j\frac{2k}{q(z)}\right]\Psi_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) - j2k\frac{\partial\Psi}{\partial z} = 0 \Rightarrow \left\{\left[\frac{k^2}{q^2(z)}\left(q'(z)-1\right)\right]r^2 - 2k\left[P'(z)+\frac{j}{q(z)}\right]r^0\right\}\Psi_0 = 0$$

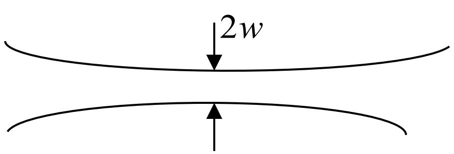
(1) $\boxed{q'(z) = 1}$ (2) $\boxed{P'(z) = -\frac{j}{q(z)}}$

$$q(z) = z + q_0 \qquad q_0 = q(z=0)$$

Introduce: $\frac{1}{q} = \frac{1}{R(z)} - \frac{j\lambda}{\pi w^2(z)}$ beam parameters

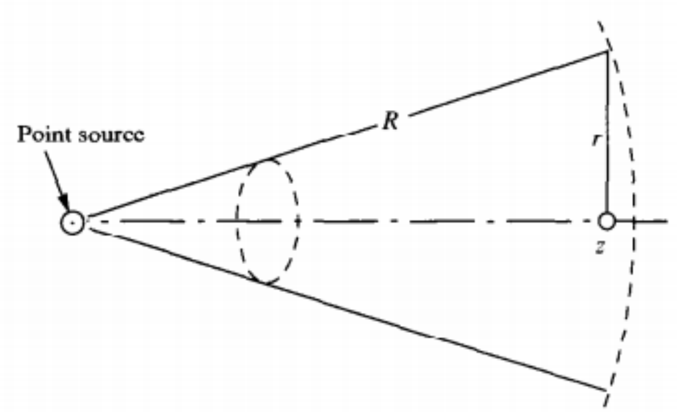
$$\Psi_0 = \exp\left\{-j\left[P(z)+\frac{kr^2}{2q}\right]\right\} = \exp\left\{-jP(z)\right\}\exp\left\{-\frac{jkr^2}{2R} - \frac{r^2}{w^2}\right\}$$

$$\Psi_0 = \exp\left\{-jP\left(z\right)\right\}\exp\left\{-\frac{jkr^2}{2R}-\frac{r^2}{w^2}\right\}$$



Spherical wave: $E \sim \frac{1}{R}e^{-jkR}$ $\quad R=\sqrt{r^2+z^2}$ $\quad$ for the region $R:\ z >> r$

$$R = z\left(1+\frac{r^2}{z^2}\right)^{\frac{1}{2}} \approx z+\frac{1}{2}\frac{r^2}{z} \Rightarrow E \sim \frac{1}{R}e^{-jkz}e^{-j\frac{kr^2}{2R}}$$



R = radius of curvature of the wavefront at z

$q = z + q_0$ $\quad \frac{1}{q} = \frac{1}{R(z)} - \frac{j\lambda}{\pi w^2\left(z\right)}$

phase $-kz - \frac{kr^2}{2R\left(z\right)} = const$

$R\left(0\right)=\infty$ $\quad$ (also by symmetry) $\quad R \sim \frac{r^2}{2z}$

For z=0 $\quad \frac{1}{q_0} = \frac{j\lambda}{\pi w^2(0)} \Rightarrow q_0 = \boxed{\frac{j\pi w_0{}^2}{\lambda} = q\left(z=0\right)}$

Determine $R(z)$ & $w(z)$

$$\frac{1}{q} = \frac{1}{R} - \frac{j\lambda}{\pi w^2} = \frac{1}{q_0 + z} = \frac{1}{\frac{j\pi w_0^2}{\lambda} + z}$$

Define $z_0 = \frac{\pi w_0^2}{\lambda}$

$$q_0 = j\frac{\pi w_0^2}{\lambda}$$

$$\frac{1}{q} = \frac{1}{R} - \frac{jw_0^2}{z_0 w^2} = \frac{1}{z + jz_0} = \frac{z - jz_0}{z^2 + z_0^2}$$

$$z_0 = \frac{\pi {w_0}^2}{\lambda}$$

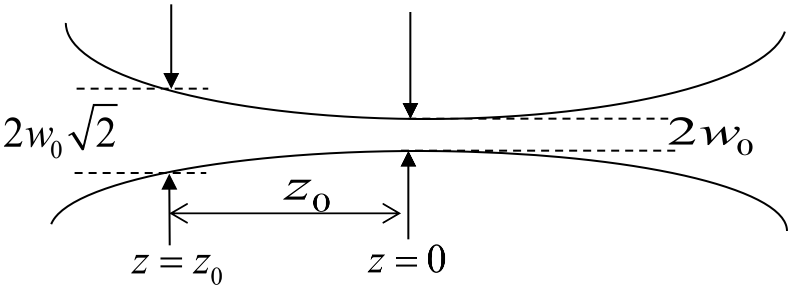
$$R = \frac{z^2 + z_0^2}{z} \qquad \frac{1}{z_0}\frac{{w_0}^2}{w^2} = \frac{z_0}{z^2 + z_0^2}$$

$$\boxed{R = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]} \quad \boxed{w^2(z) = {w_0}^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]} \qquad w^2(z) = {w_0}^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]$$

(1) z_0 defines the length of the focal region

when $z = z_0$ $\quad w^2 = 2{w_0}^2$

z_0 − confocal parameter



(2) Beam radius $w(z)$

$$E \sim e^{\frac{-r^2}{w^2}}$$

(3)

$$R = \infty \quad \text{at focus} \quad (z = 0)$$
$$R = 2z_0 \quad \text{at} \quad z = z_0$$
$$R = z \quad \text{at} \quad z >> z_0$$

divergence angle $=\frac{\theta}{2}=\lim_{z>>z_0}\frac{dw}{dz}=\frac{w_0}{z_0}=\frac{\lambda}{\pi w_0}=\frac{\theta}{2}$

$$\uparrow$$

$$\frac{dw}{dz}=w_0\frac{1}{2}\frac{1}{\sqrt{1+\frac{z^2}{z_0{}^2}}}\cdot 2\frac{z}{z_0^2}\Rightarrow\frac{w_0}{z_0}$$

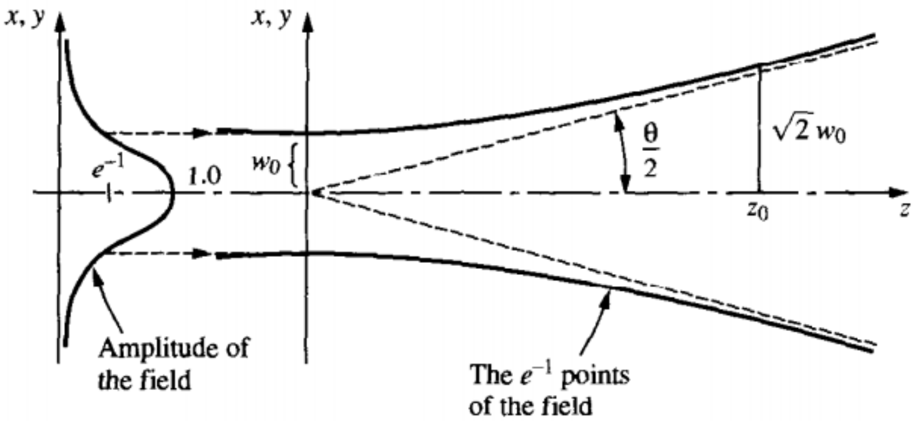


FIGURE 3.2. Spreading of a $TEM_{0,0}$ mode.

Find $P(z)$

$$P'(z)=\frac{-j}{q(z)}=\frac{j}{z+jz_0}$$

$$\int_0^z\frac{dz'}{z'+jz_0}=ln\left(z'+jz_0\right)\Big|_0^z=ln\left(z+jz_0\right)-ln\left(jz_0\right)$$

$$jP(z)=ln\left[1-j\left(\frac{z}{z_0}\right)\right]$$

using: $a - jb = \sqrt{a^2 + b^2}\,\exp\left[-j\tan^{-1}\frac{b}{a}\right]$

$$1 - j\frac{z}{z_0} = \sqrt{1 + \left(\frac{z}{z_0}\right)^2}\,\exp\left[-j\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$

$$jP(z) = ln\left(1 - j\frac{z}{z_0}\right) = ln\sqrt{1 + \left(\frac{z}{z_0}\right)^2} - j\tan^{-1}\frac{z}{z_0}$$

$$\exp\left[-jP(z)\right] = \exp\left[-ln\sqrt{1 + \left(\frac{z}{z_0}\right)^2} + j\tan^{-1}\frac{z}{z_0}\right]$$

$$= \left[1 + \left(\frac{z}{z_0}\right)^2\right]^{-1/2} \exp\left(j\tan^{-1}\frac{z}{z_0}\right)$$

$$E(x,y,z) = E_0 \underbrace{\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{-1/2}}_{\frac{w_0}{w}} \exp\left(-\frac{r^2}{w^2(z)}\right)\exp\left\{\underbrace{-jkz}_{\text{Plane wave}} + \underbrace{j\tan^{-1}\frac{z}{z_0}}_{\text{π change on going thru focus}} \underbrace{- j\frac{kr^2}{2R}}_{\text{Radial phase vector}}\right\}$$

Field amplitude decreases with r and with z

At z_o field amplitude $\frac{1}{\sqrt{2}}$ on axis ; spot size increases by $\sqrt{2}$

Total Power

$$= \frac{1}{2}\int E \times H^{*} dA = \frac{1}{2}\int \frac{EE^{*}}{\eta} dA = \frac{1}{2}\frac{E_0^2}{\eta}\int \left(\frac{w_0}{w(z)}\right)^2 \exp\left(\frac{-2r^2}{w^2(z)}\right) \cdot r dr d\varphi$$

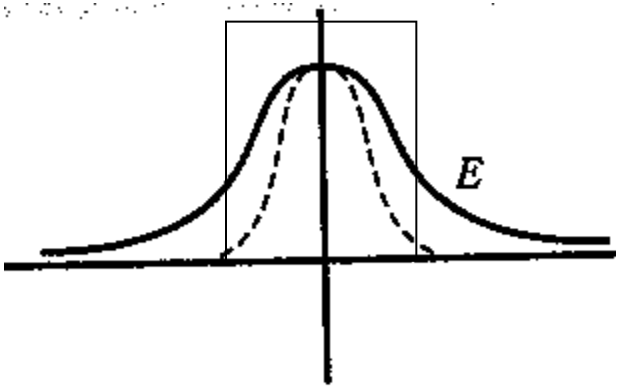
effective area

$$= \frac{1}{2}\frac{E_0^2}{\eta}\underbrace{\left[\frac{\pi w_0^2}{2}\right]}_{\uparrow}$$

effective area

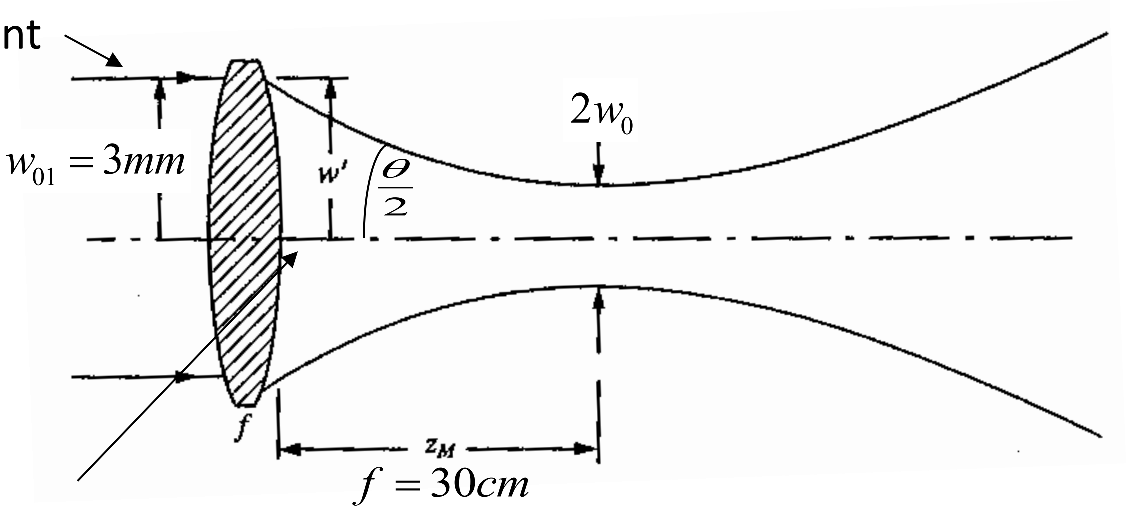
vacuum impedance

$$\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$



Example

Plane wave front



$P = 30mW$

$\lambda = 1.064\mu m$

find $\mathrm{w}_{02}, z_{02}, \mathrm{I}_{\max}, E_{\max}$

$$\frac{\theta}{2} \cong \frac{w_{01}}{f} = \frac{3mm}{30cm} = 10^{-2} = \frac{\lambda}{\pi w_{02}} \qquad \left(\frac{\theta}{2} = \frac{w_0}{z_0} = \frac{\lambda}{\pi w_0}\right)$$

$$w_{02} = \frac{\lambda}{\pi\left(\theta/2\right)} = \frac{1.064\mu m}{\pi\left(10^{-2}\right)} = 33.9\mu m$$

$$z_{02} \overset{df}{=} \frac{\pi w_{02}^2}{\lambda} = \frac{w_{02}}{\theta/2} = \frac{33.9\mu m}{10^{-2}} = 3.4mm \qquad \left(\frac{\theta}{2} = \frac{w_{02}\pi}{\lambda}\right)$$

$$\mathrm{I}_{\max} = \frac{P}{\left(\frac{\pi w_{02}^2}{2}\right)} = \frac{30mW}{\frac{\pi\left(33.9\mu m\right)^2}{2}} = 1.7\times10^3\,\frac{W}{cm^2}$$

$$E_{\max} = \sqrt{2\eta I} = \sqrt{2\left(377\Omega\right)\left(1.7\times10^3\,\frac{W}{cm^2}\right)} = 1.1\times10^3\,\frac{V}{cm} = 1.1\times10^5\,\frac{V}{m}$$