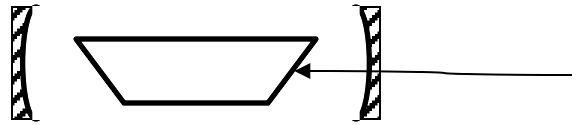
Higher Order Modes



Windows break the cylindrical symmetry

$$E(x,y,z) = E_{m,p} H_m\left[\frac{\sqrt{2}x}{w(z)}\right] H_p\left[\frac{\sqrt{2}y}{w(z)}\right] \times \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right]$$
$$\times \exp\left\{-j\left[kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)+\frac{kr^2}{2R(z)}\right]\right\}$$

Hermite polynomials: $H_m(u) = (-1)^m e^{u^2} \dfrac{d^m e^{-u^2}}{du^m}$

$H_0(u) = 1 \quad H_1(u) = 2(u) \quad H_2(u) = 4u^2 - 2 \quad H_3(u) = 8u^3 - 12u$

m = # of zeros in x direction; p= # of zeros in y direction

$w(z), R(z)$ – are the same as for lowest order mode

$\phi = (1+m+p)\tan^{-1}\left(\dfrac{z}{z_0}\right)$ Phase inversion on going through focus

Higher Order Modes

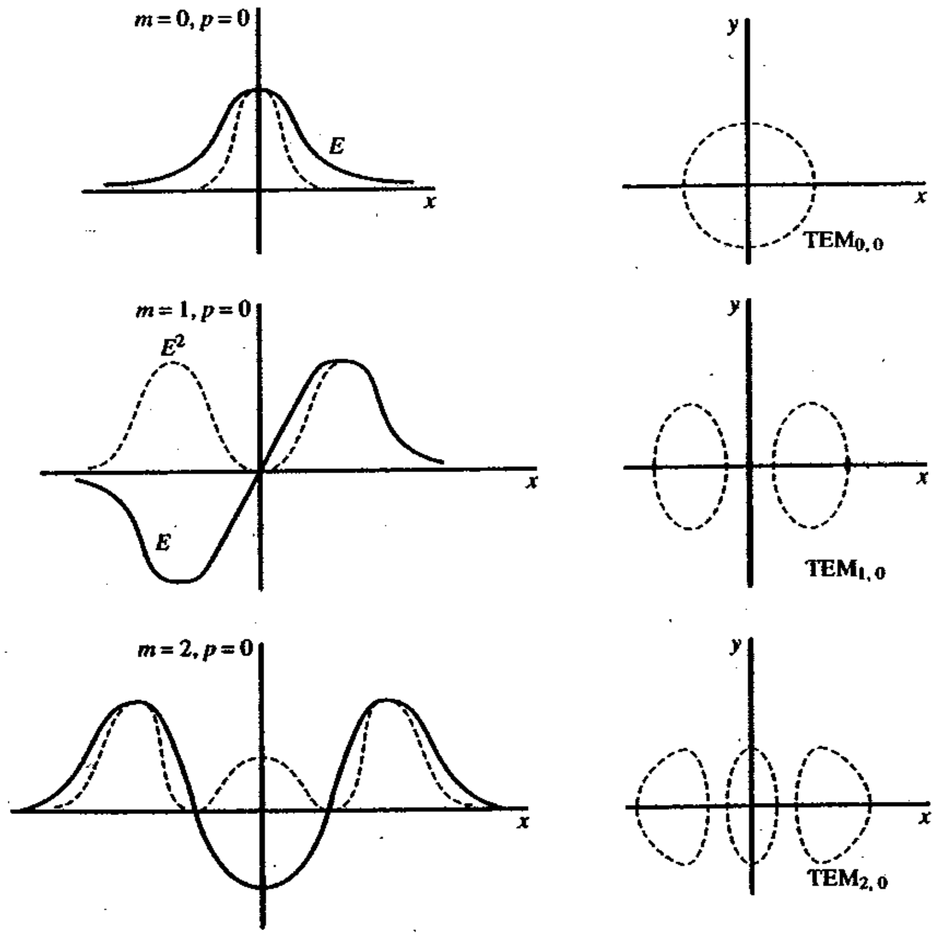


FIGURE 3.5. The field E, intensity E^2, and "dot" pattern of various modes.

ABCD Law of Gaussian Beams

The ABCD law relates beam parameters, q_2, of a Gaussian beam at plane 2 to the value q_1 at plane 1 by using the ABCD ray matrix (no strict proof but it works!):

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \qquad (1)$$

Example:

$q'(z) = 1 \Rightarrow q_2 = q_1 + z$ For free space of length z $T = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$

$\left[\Psi_0 \sim e^{-j\frac{k r^2}{2q}} \right]$ $from(1) \Rightarrow q_2 = q_1 + z$ (Same results)

Reciprocal form: $\frac{1}{q} = \frac{1}{R} - j\frac{\lambda_0}{\pi n w^2}$ (n – refractive index) $\lambda = \frac{\lambda_0}{n}$

From (1): $\frac{1}{q_2} = \frac{C + D\left(\frac{1}{q_1}\right)}{A + B\left(\frac{1}{q_1}\right)}$

ABCD law

$$\boxed{\frac{1}{q_2} = \frac{C + D\left(\frac{1}{q_1}\right)}{A + B\left(\frac{1}{q_1}\right)}} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}$$

Example: (a beam at focus: w_0 & planar front at z=0)

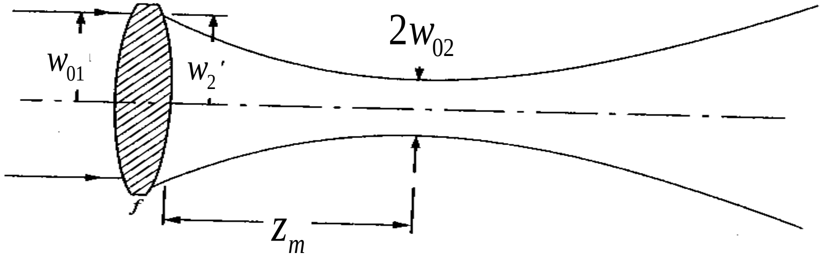
$$\frac{1}{q_2(z)} = \frac{0 + 1\cdot\left(-j\lambda/\pi w_0^{\ 2}\right)}{1 + z\left(-j\lambda/\pi w_0^{\ 2}\right)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

(reproduce the expansion law for a Gaussian Beam)

$$\Rightarrow R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]; w^2(z) = w_0^{\ 2}\left(1 + \left(\frac{\lambda z}{\pi w_0^{\ 2}}\right)^2\right)$$

<u>Example</u>: a beam transformation by a lens

a planer wave front impinges on a lens



to the right to the lens (right behind):

$$\frac{1}{q_2}=\frac{-\frac{1}{f}+1\cdot\left(\frac{1}{q_1}\right)}{1-0\cdot\left(\frac{1}{q_1}\right)}=-\frac{1}{f}+\frac{1}{q_1} \qquad T=\begin{pmatrix}1 & 0\\ -\frac{1}{f} & 1\end{pmatrix}$$

$$\frac{1}{q_1}=\frac{1}{\infty}-j\frac{\lambda_0}{\pi w_{01}^2}$$

$$\left|R_2\right|=f$$

$$\Rightarrow\left(\frac{1}{q_2}\overset{df}{=}\frac{1}{R_2}-j\frac{\lambda}{\pi {w_2}^2}\right)=-\frac{1}{f}-j\frac{\lambda_0}{\pi w_{01}^2}\Rightarrow\boxed{w_2=w_{01}} \qquad (\lambda=\lambda_0)$$

Same Radius!

For a lens + z of free space

$$\frac{1}{q_3}=\frac{1}{R(z)}-j\frac{\lambda}{\pi w^2(z)}=\frac{C+D\frac{1}{q_1}}{A+B\frac{1}{q_1}}= \qquad \mathbf{T}=\begin{pmatrix}1-\frac{z}{f} & z\\ -\frac{1}{f} & 1\end{pmatrix}$$

$$\frac{1}{q_1}=-j\frac{\lambda}{\pi w_{01}^2}=-\frac{j}{z_{01}}$$

$$\frac{1}{q_3}=-\frac{\frac{1}{f}+\frac{j}{z_{01}}}{\left(1-\frac{z_m}{f}\right)-j\frac{z_m}{z_{01}}}=-\frac{\left[\frac{1}{f}\left(1-\frac{z_m}{f}\right)-\frac{z_m}{z_{01}^2}\right]+j\left[\frac{z_m}{z_{01}f}+\frac{1}{z_0}\left(1-\frac{z_m}{f}\right)\right]}{\left(1-\frac{z_m}{f}\right)^2+\left(\frac{z_m}{z_{01}}\right)^2}$$

m – "minimum"

$$\frac{1}{R} = 0 \Rightarrow \frac{1}{f}\left(1 - \frac{z_m}{f}\right) - \frac{z_m}{z_{01}^2} = 0$$ (Real part = 0)

$$z_m\left(-\frac{1}{f^2} - \frac{1}{z_{01}^2}\right) + \frac{1}{f} = 0 \Rightarrow \boxed{z_m = \frac{f}{1 + \frac{f^2}{z_{01}^2}} \approx f}$$

$$\left(z_{01} = \frac{\pi w_{01}{}^2}{\lambda} >> f\right)$$

$$\frac{\lambda}{\pi w_{03}^2} \overset{z_m \approx f}{=} \frac{\frac{1}{z_{01}}}{\left(1 - \frac{z_m}{f}\right)^2 + \left(\frac{z_m}{z_{01}}\right)^2} \overset{z_m \approx f}{\approx} \frac{z_{01}}{f^2}$$

$$z_{01} = \frac{\pi w_{01}^2}{\lambda} \Rightarrow \frac{\lambda}{\pi w_{03}^2} = \frac{\pi w_{01}^2}{\lambda f^2} \Rightarrow \boxed{w_{03} = \frac{f\lambda}{\pi w_{01}}}$$

$$(f << z_{01})$$

<u>Note</u>: given q (or q^{-1}) at some point,

$-\mathrm{Re}(q)$ = focus position

$+\mathrm{Im}(q) = z_0$

$$q = z + jz_0$$

$$\left(\frac{1}{q} = \frac{1}{R} - \frac{j\lambda}{\pi w_0^2}; \text{At focus}: R = \infty \Rightarrow \mathrm{Re}(q) = 0\right)$$

1) Focus position is at $-\mathrm{Re}(q)$ from a given point

2) $+\mathrm{Im}(q) = z_0$

Cascading the ABCD Law

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1} \qquad q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2}$$

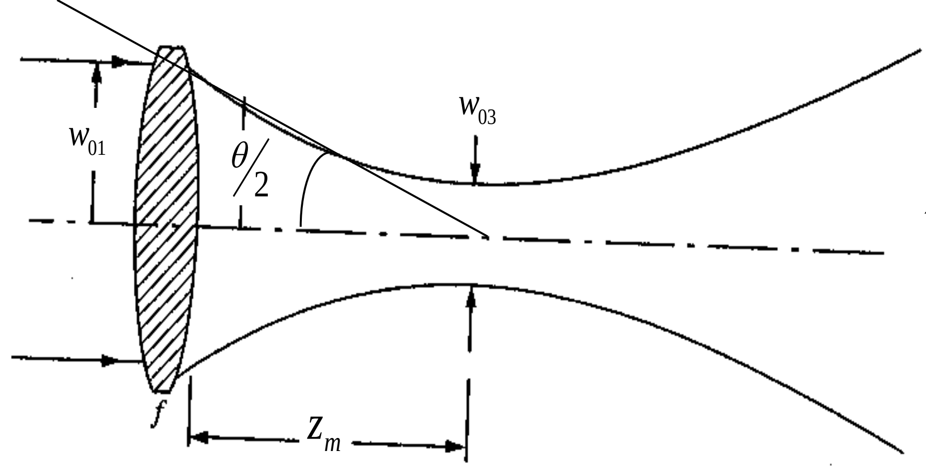
Combine by substituting q_2 into q_3

$$q_3 = \overset{\text{Algebra}}{\ldots\ldots\ldots\ldots\ldots} = \frac{A q_1 + B}{C q_1 + D}$$

Where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$

$$\mathrm{T} = \mathrm{T}_2\, \mathrm{T}_1$$

Example Revisited:



$\lambda = 1.06\mu m$

$f = 30cm$

$w_{01} = 3mm$

Find w_{03} and its location

$$\frac{1}{q_1} = \frac{1}{R_1} - \frac{j\lambda}{\pi w_{01}^2} = 0 - \frac{j\,1.06\mu m}{\pi (3mm)^2} = -j3.763\times10^{-4}\,cm^{-1}$$

Behind the lens:

$$T = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \qquad \frac{1}{q_2} = \frac{C + D\left(\frac{1}{q_1}\right)}{A + B\left(\frac{1}{q_1}\right)} = \frac{-\frac{1}{f} + 1\left(\frac{1}{q_1}\right)}{1 + 0\left(\frac{1}{q_1}\right)} = \frac{1}{q_1} - \frac{1}{f} = -j\left(3.763\times10^{-4}\,cm^{-1}\right) - 0.03333cm^{-1}$$

$$\left.\begin{aligned} \frac{1}{q} &= a + ib \\ q &= \frac{a - ib}{a^2 + b^2} \end{aligned}\right\} \qquad q_2 = \left(-29.999 + j0.3387\right)cm = z + jz_0$$

Where is the focus?

$$R = \infty \Rightarrow \mathrm{Re}(q) = 0 \qquad \mathrm{Im}(q_2) = \frac{\pi w_0^2}{\lambda} = z_0 = \mathrm{Im}(q_3) = 0.3387cm$$

$$q_3 = q_2 + \underset{\boxed{-\mathrm{Re}\,q_2}}{\underset{\parallel}{\Delta z}} \Rightarrow \Delta z \Rightarrow 29.999cm$$

(must be purely imaginary)

$$\frac{1}{q_3} = -\frac{j\lambda}{\pi w_{03}^2}$$

$R = \infty$ (purely imaginary)

$q_3 = j \operatorname{Im} q_3$ at focus

$$w_{03} = \sqrt{\frac{-jq_3\lambda}{\pi}} = \sqrt{\frac{(0.3387cm)(1.064\mu m)}{\pi}} = 33.9\mu m$$

(same results)

Lets remove the assumption that the incoming beam had a planar wave front (i.e. $R \neq \infty$)

$$\frac{1}{q_2} = \frac{C + D\left(1/q_1\right)}{A + B\left(1/q_1\right)} \qquad \frac{1}{q_1} = \frac{1}{R_1} - j\frac{\lambda_0}{\pi w_1^2}$$

Lens: $C = \frac{-1}{f}$ A=1 B=0 D=1

$$\Rightarrow \frac{1}{q_2} = -\frac{1}{f} + \frac{1}{R_1} - j\frac{\lambda}{\pi w_1^2} \overset{df}{=} \frac{1}{R_2} - j\frac{\lambda}{\pi w_2^{\ 2}} \Rightarrow w_2 = w_1$$

Thin lens keeps the spot size the same! Therefore conserves the power, but it changes the radius of curvature of the incoming beam to $\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$ (thin lens formula)

$$w_1 = w_2$$

Lens: $$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$