Optical Cavities

Cavity provides the feedback system needed for laser operation!

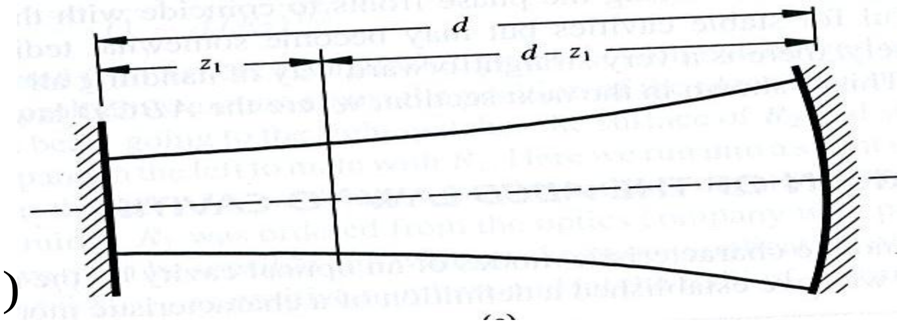
A cavity mode is a field distribution that reproduces itself in relative shape and in relative phase after a round trip.

To follow a Gaussian beam through a round trip in a cavity we employ the ABCD law.

1. We assume that the Hermite-Gaussian beams are the characteristic modes of the optical cavity (Mirrors exactly match the surfaces of constant phase of the beams.)

For this require that:
2. The complex beam parameter repeats itself after a round trip!

$$q(z_1 + \mathrm{roundtrip}) = q(z_1)$$



$$q(z_1 + \text{roundtrip}) = q(z_1)$$

$$q(z_1) = \frac{Aq(z_1) + B}{Cq(z_1) + D}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

transmission matrix
for the unit cell
(round trip)

$$Cq^2 + Dq = Aq + B$$

$$B\left(\frac{1}{q}\right)^2 + 2\left(\frac{A-D}{2}\right)\left(\frac{1}{q}\right) - C = 0$$

$$\frac{1}{q} = -\frac{A-D}{2B} \pm \frac{1}{B}\left[\left(\frac{A-D}{2}\right)^2 + BC\right]^{\frac{1}{2}}$$

$$AD - BC = 1 \qquad \frac{A^2}{4} - \frac{AD}{2} + \frac{D^2}{4} + BC\left(1 - \frac{AD}{2} + \frac{AD}{2}\right) = \left(\frac{A+D}{2}\right)^2 - 1$$

$$\frac{1}{q} = -\frac{A-D}{2B} - j\frac{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}}{B}$$

$$\frac{1}{q} = \frac{1}{R} - j\,\frac{\lambda}{\pi w^2}$$

$$R(z_1) = -\frac{2B}{A-D} \qquad \frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}}$$

$$\mathrm{T} = \begin{pmatrix} 1 & d+z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d-z_1 \\ -\frac{1}{f} & 1-\frac{d-z_1}{f} \end{pmatrix}$$

If we start the unit cell at the flat mirror $(z=0)$

$$\mathrm{T} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ -\frac{1}{f} & 1-\frac{d}{f} \end{pmatrix} = \begin{pmatrix} 1-\frac{d}{f} & d+d\left(1-\frac{d}{f}\right) \\ 0 & 1-\frac{d}{f} \end{pmatrix} \qquad (R_L = \infty)$$

$$\frac{\pi w_0{}^2}{\lambda} = \frac{d+d\left(1-\frac{d}{f}\right)}{\left[1-\left(1-\frac{d}{f}\right)^2\right]^{\frac{1}{2}}} = \frac{2d\left(1-\frac{d}{R}\right)}{\left[4\left(\frac{d}{R}\right)\left(1-\frac{d}{R}\right)\right]^{\frac{1}{2}}} \qquad f = \frac{R}{2}$$

$$\boxed{\frac{\pi w_0{}^2}{\lambda} = (dR)^{\frac{1}{2}}\left(1-\frac{d}{R}\right)^{\frac{1}{2}}}$$

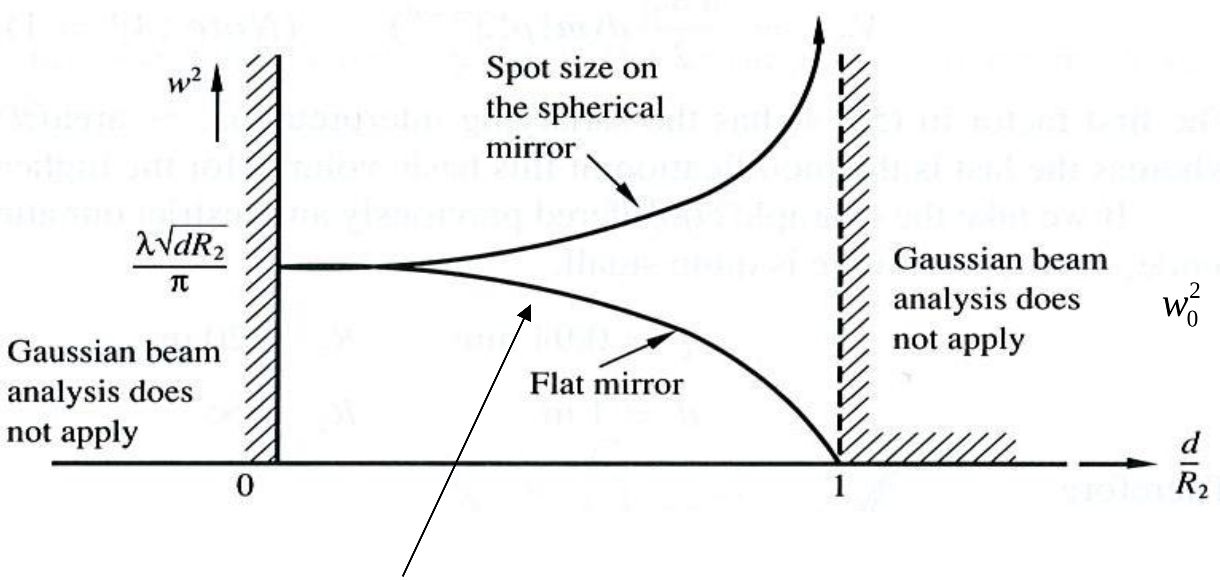
$$w^2(z) = w_0^2\left[1+\left(\frac{z}{z_0}\right)^2\right]$$

z_0^2 – confocal parameter

$$z_0^2 = \left(\frac{\pi {w_0}^2}{\lambda}\right)^2$$

$$\frac{\pi w^2(d)}{\lambda} = (dR)^{\frac{1}{2}}\left(1-\frac{d}{R}\right)^{\frac{1}{2}}\left[1+\frac{d^2}{dR\left(1-\frac{d}{R}\right)}\right] = \frac{(dR)^{\frac{1}{2}}}{\left(1-\frac{d}{R}\right)^{\frac{1}{2}}}$$

On a spherical mirror

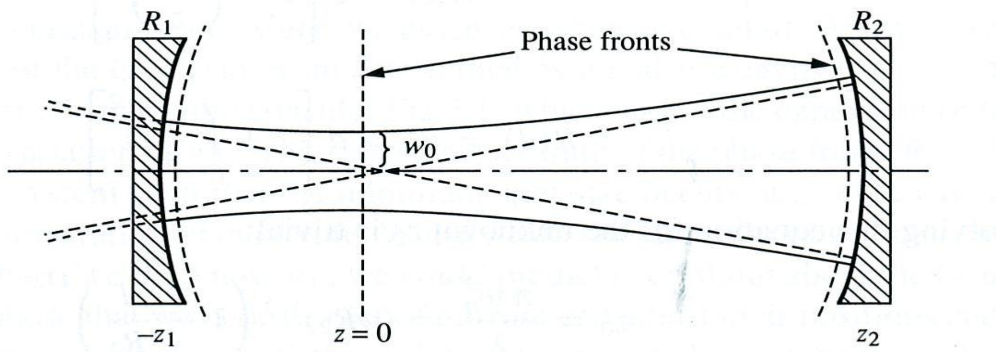


(not physical)

w_0^2 becomes imaginary

$$\frac{\pi w_0^2}{\lambda} = (dR)^{\frac{1}{2}}\left(1-\frac{d}{R}\right)^{\frac{1}{2}}$$

The requirement that mirrors must match the surface of the constant phase (so that a cavity mode is excited) allows one to find where the mirrors with R_1 & R_2 should be placed, in general case.



R_1 , R_2 are given, we look for z_1 z_2

1) $z_1 + z_2 = d$ 2) $R(z_2) = R_2 = z_2\left[1+\left(\frac{z_0}{z_2}\right)^2\right]$ 3) $R(z_1) = -R_1 = -z_1\left[1+\left(\frac{z_0}{z_1}\right)^2\right]$

The wave front on the left at z=0 has a (mathematically) negative radius of curvature, but we know that the mirror R_1 has positive (focusing) properties. We treat all distances z_1 z_2 as positive numbers and let the radii of curvature of the mirrors carry their own sign.

Solving (involving algebra)….

$$z_0^2 = \left(\frac{\pi w_0^{\ 2}}{\lambda}\right)^2 = \frac{d(R_1-d)(R_2-d)(R_1+R_2-d)}{(R_1+R_2-2d)^2}$$

$$z_1 = \frac{d(R_2-d)}{R_1+R_2-2d} \qquad z_2 = \frac{d(R_1-d)}{R_1+R_2-2d}$$

$$R_1 = \infty \qquad z_0^2 = d(R_2-d) \Rightarrow dR_2\left(1-\frac{d}{R_2}\right) \qquad z_1 = 0; z_2 = d \qquad \frac{\pi w_0^2}{\lambda} = (dR)^{\frac{1}{2}}\left(1-\frac{d}{R}\right)^{\frac{1}{2}}$$

Same result

Summary

1) Postulate that Hermite-Gaussian Beams are the normal modes for the cavity.

2) Formulate an equivalent transmission system for this cavity showing one round trip. Identify a unit cell.

3) Force the complex beam parameter to transform into itself after a round trip by use of the ABCD law.

4) Evaluate R and w using:

$$R(z) = -\frac{2B}{A-D} \qquad \frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}}$$

The theory applies for Stable cavity only!

Mode Volume

$$E_0^2 V = \int_0^d dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, E(x,y,z) E^*(x,y,z)$$

$$E_0^2 V_{m,n} = E_0^2 \int_0^{\infty} \frac{w_0^2}{w^2(z)} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; H_m^2\left(\frac{\sqrt{2}x}{w}\right) e^{-\frac{2x^2}{w^2}} \times H_n^2\left(\frac{\sqrt{2}y}{w}\right) e^{-\frac{2y^2}{w^2}}$$

use $u = \frac{\sqrt{2}x}{w} \, or \, \frac{\sqrt{2}y}{w}$

$$V_{m,n} = \int_o^d \frac{{w_0}^2}{2} dz \left[\int_{-\infty}^{-\infty} {H_m}^2(u) e^{-u^2} du\right]\left[\int_{-\infty}^{\infty} {H_n}^2(u) e^{-u^2} du\right]$$

$$\underbrace{\int_{-\infty}^{\infty} {H_n}^2(u) e^{-u^2} du}_{2^n n! \sqrt{\pi}}$$

$$V_{m,n} = \underbrace{\frac{\pi {w_0}^2}{2}}_{\text{Area x length}} \underbrace{d\left(m!n!2^{m+n}\right)}_{\text{Modification for high-order modes}}$$

Area x length — Modification for high-order modes

Example (textbook)

$w_0 = 0.94 mm$ $R_2 = 20\ m$

$d = 1\ m$ $R_1 = \infty$

$$\rightarrow V_{0.0} = 1.38 cm^3 = \frac{\pi w_0{}^2}{2} \times d$$

He-Ne laser

$P = 0.1\ torr$ (of neon)

Each atom is excited (by the gas discharge) and thus producing a photon at 632.8 µm, say, 10 times per second.

$$\text{Energy per photon } hv = \frac{hc}{\lambda} = 3.14 \times 10^{-19} J = 1.96 eV$$

$$\times \text{ \# of Ne atoms } = 0.1(3.54 \times 10^{16}) V_{0,0} = 4.88 \times 10^{15}$$

$$\times \text{ (average excitation per atom = average emission per atom) } = 10\ \sec^{-1}$$

$$= \text{ Power } = 15.3 mW$$

typical!