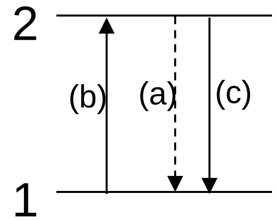
Einstein's A & B coefficients

2

2- level
system



a) Spontaneous emission

$$\left.\frac{dN_2}{dt}\right|_{SpE} = -A_{21}N_2$$

N_2 = population density of state 2

$\tau = \left(A_{21}\right)^{-1}$ = decay lifetime from state 2 to 1

b) Absorption

$$\left.\frac{dN_2}{dt}\right|_{A} = B_{12}N_1\rho(v)$$

N_1 = population density of state 1

$\rho(v)$ = energy density

c) Stimulated emission

$$\left.\frac{dN_2}{dt}\right|_{StE} = -\mathrm{B}_{21}\mathrm{N}_2\rho(v)$$

At equilibrium:

$$\frac{dN_2}{dt} = -A_{21}N_2 + \mathrm{B}_{12}\mathrm{N}_1\rho(v) - \mathrm{B}_{21}\mathrm{N}_2\rho(v) = 0$$

$$N_2\left[-A_{21}-B_{21}\rho(v)\right]+N_1\left[B_{12}\rho(v)\right]=0$$

$$\boxed{\frac{N_2}{N_1}=\frac{B_{12}\rho(v)}{A_{21}+B_{21}\rho(v)}}$$

Degeneracy of state (g): # of different ways atom can have the same energy

$$\frac{N_2}{N_1}=\frac{g_2}{g_1}\exp\left[-hv/kT\right]$$

hv = the energy difference between the two atomic states

$$\frac{B_{12}\rho(v)}{A_{21}+B_{21}\rho(v)}=\frac{g_2}{g_1}\exp\left[-hv/kT\right]$$

for extreme temps: $\exp\left[-hv/kT\right]\sim 1$ and large ρ (stimulated process dominates)

$$\boxed{\frac{B_{12}}{B_{21}}=\frac{g_2}{g_1}}$$

Now we find $\frac{A_{21}}{B_{21}}$ by using $B_{12}=\frac{g_2}{g_1}B_{21}$ and $\rho(v)=\frac{8\pi v^3 n_r^3}{c^3}\frac{hv}{e^{-hv/kT}-1}$

$$\frac{g_2}{g_1}B_{21}\rho(v)=\frac{g_2}{g_1}A_{21}e^{-hv/kT}+\frac{g_2}{g_1}B_{21}e^{-hv/kT}\rho(v)$$

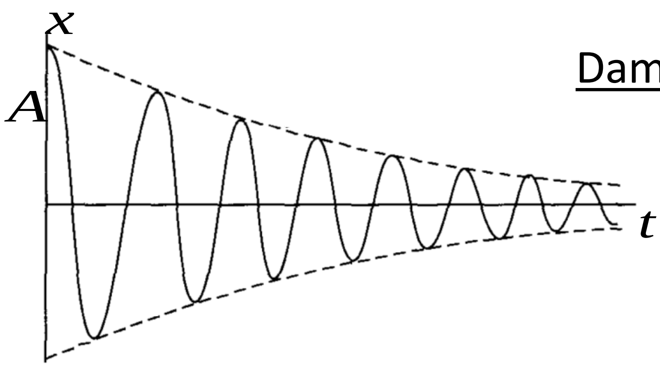
$$B_{21}\left(1-e^{hv/kT}\right)\rho(v)=A_{21}e^{-hv/kT}$$

$$B_{21} e^{-h\nu/kT} \left(e^{h\nu/kT} - 1 \right) \frac{8\pi h \nu^3 n_r^3}{c^3} \left(\frac{1}{e^{h\nu/kT} - 1} \right) = A_{21} e^{-h\nu/kT}$$

$$\boxed{\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3 n_r^3}{c^3}}$$

- Not limited to equilibrium despite the fact that we used this

Line Shape



Damped Free Harmonic Oscillator

$$-kx - m\gamma\dot{x} = m\ddot{x}$$

$$m\ddot{x} + m\gamma\dot{x} + kx = 0 \qquad \omega_0^2 = \frac{k}{m}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$$x = \mathrm{A}e^{-\gamma' t/2} e^{-j\omega t}$$

A is complex (includes phase)

$$\left(-\frac{\gamma'}{2} - j\omega\right)^2 + \gamma\left(-\frac{\gamma'}{2} - j\omega\right) + \omega_0^2 = 0$$

$$\frac{\gamma'^2}{4} + j\omega\gamma' - \omega^2 + \gamma\left(-\frac{\gamma'}{2} - j\omega\right) + \omega_0^2 = 0$$

equate imaginary parts: $\gamma' = \gamma$

equate real parts: $-\dfrac{\gamma^2}{4} - \omega^2 + \omega_0^2 = 0$ $\qquad \omega = \left[\omega_0^2 - \dfrac{\gamma^2}{4}\right]^{1/2} = \omega_0\left[1 - \dfrac{\gamma^2}{4\omega_0^2}\right]^{1/2}$

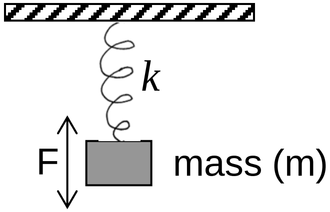
$$\boxed{x = \mathrm{A}e^{\frac{-\gamma t}{2}} e^{-j\left(\omega_0^2 - \frac{\gamma^2}{8\omega_0}\right)t}} \qquad \boxed{\omega \approx \omega_0^2 - \frac{\gamma^2}{8\omega_0}}$$

1) For the free damped oscillator, $\gamma/2$ is the decay rate of the damped oscillation.

2) For the driven oscillator (see below), γ gives the bandwidth of the resonant response.

The analogy carries over to the atomic decay!

Damped Driven Harmonic Oscillator



$$\Sigma F = -kx - m\gamma\dot{x} + q\mathrm{E}\cos\omega t = ma$$

$$m\ddot{x} + m\gamma\dot{x} + kx = q\mathrm{E}\cos\omega t$$

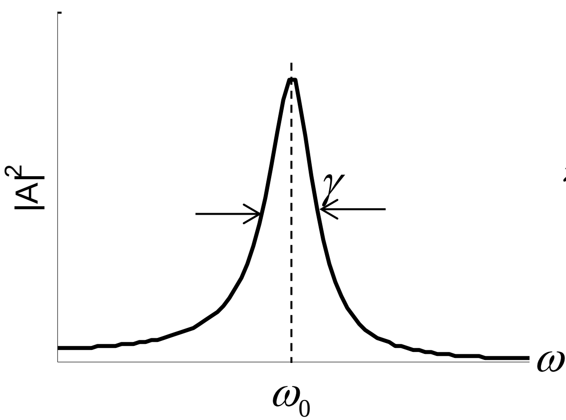
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{q}{m}\mathrm{E}\cos\omega t$$

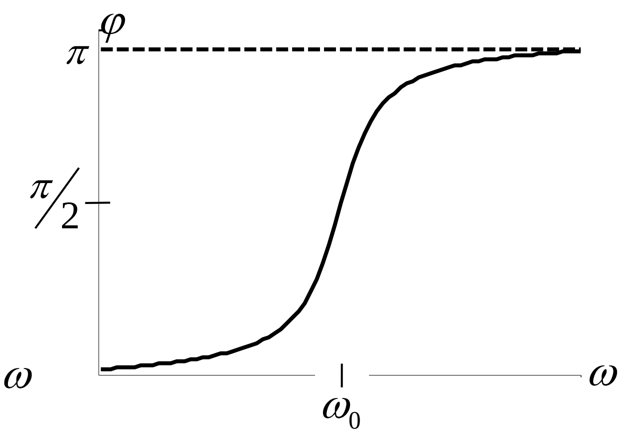
Search solution in the form: $x = \mathrm{A}e^{-j\omega t}$

$$\frac{q}{m}Ee^{-j\omega t} = \left[-\omega^2 - \gamma j\omega + \omega_0^2\right]Ae^{-jwt}$$

"actual" solution is given by real part

$$A = \frac{{}^{q}\!/_{m}\,E}{\left(\omega_0^2 - \omega^2 - j\gamma\omega\right)} \qquad |A| = \frac{{}^{q}\!/_{m}\,E}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2\right]^{1/2}} \qquad |A|^2 = \frac{\left({}^{q}\!/_{m}\right)^2 |E|^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2}$$





$$\varphi = \tan^{-1}\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)}$$

When: $\gamma << \omega_0 \Rightarrow \quad |\omega - \omega_0| << \omega_0 \Rightarrow \quad \omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) \approx 2\omega_0(\omega_0 - \omega)$

$$\boxed{|\mathrm{A}|^2 = \left(\frac{q\mathrm{E}}{2m\omega}\right)^2 \frac{1}{(\omega_0 - \omega)^2 + \left({}^{\gamma}\!/_{2}\right)^2}}$$

$$\varphi = \tan^{-1}\frac{{}^{\gamma}\!/_{2}}{\omega_0 - \omega}$$