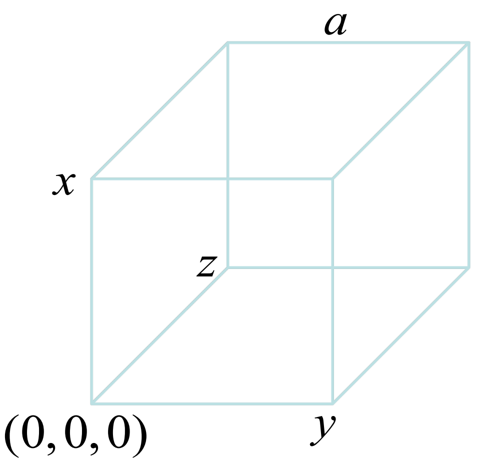
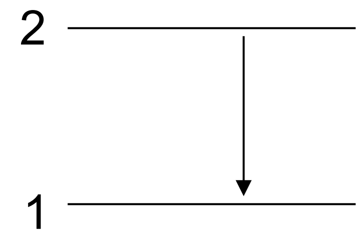
Quantization of EM Field

$$E_n = nh\nu$$ - energy in mode n

Plank vs. ("Ultraviolet catastrophe") blackbody radiation theory



Equilibrium between blackbody emitters and radiation field



Quantified atomic energy levels [absorption, spontaneous emission, stimulated emission]

$$\nabla^2 \mathrm{E} + k^2 \mathrm{E} = 0 \qquad \mathrm{E} = \mathrm{E}_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)$$

$$\begin{aligned} \mathrm{E} = 0 \text{ at } x &= 0, a & & k_x a = m\pi \\ y &= 0, a & \Rightarrow \quad & k_y a = n\pi \quad \Rightarrow \quad k^2 = k_x{}^2 + k_y{}^2 + k_z{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{q\pi}{a}\right)^2 = \left(\frac{2\pi n_r \nu}{c}\right)^2 \\ z &= 0, a & & k_z a = q\pi \end{aligned}$$

Quantized k!

$$\vec{k} = \frac{\pi}{a}\left(m\hat{x} + n\hat{y} + q\hat{z}\right)$$

$$\nu_{m,n,q} = \frac{c}{2n_r a}\left[m^2 + n^2 + q^2\right]^{\frac{1}{2}}$$

Quantized ν !

1. Cannot have a mode with 2 zeros.

example: (m,0,0)

$E_y = E_0 \sin k_x x$ violates $E_T = 0$ at (x, y) plane (for $x > 0$)

$E_z = E_0 \sin k_x x$ violates $E_T = 0$ at (x, z) plane

2. For a mode with one zero, only one polarization is allowed.

example: (m,n,0) $E_z = E_0 \sin(k_x x)\sin(k_y y)$ but not $\mathrm{E}_x, \mathrm{E}_y$

$$\nu_{m,n,q} = \underbrace{\frac{c}{2n_r a}}_{\text{Scale factor}} \underbrace{\left[m^2 + n^2 + q^2\right]^{\frac{1}{2}}}_{\text{Radius in k-Space}}$$

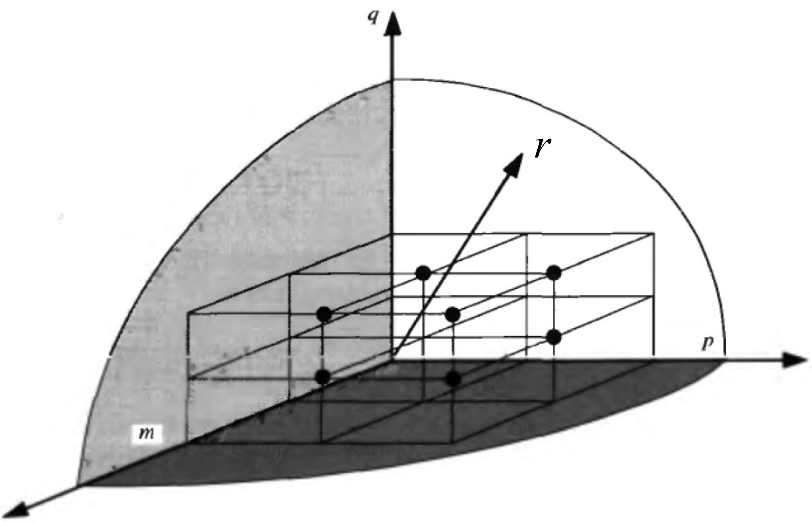


FIGURE 7.3. The mode diagram for cubic cavity.

$$r = \frac{2n_r a\nu}{c}$$

# of modes with frequency $0-\nu$

= volume of one octant of the sphere of radius r X2 independent polarizations (TE,TM)

each point represents two independent modes: TE and TM

$$N = \frac{1}{8} \times \frac{4\pi r^3}{3} \times 2 = \frac{\pi r^3}{3} = \boxed{N = \frac{8\pi a^3 \nu^3 n_r^3}{3c^3}}$$

Mode Density

(# of modes/frequency/volume) =

$$n(\nu) = \frac{1}{V}\frac{dN}{d\nu} = \frac{8\pi n_r^3 \nu^2}{c^3}$$

If EM field energy is not quantized (classiscal) - Boltzmann's statistics*
probability of energy between E and $E + dE$ is

$$dP = c \exp(-E / \mathrm{k}T)\, dE$$ - allows all energies (classical)

$$\langle E \rangle = \frac{\int E dP}{\int dP} = \frac{\int_0^\infty E \exp(-E/\mathrm{k}T)\, dE}{\int_0^\infty \exp(-E/\mathrm{k}T)\, dE} = \mathrm{k}T$$ (Boltzmann)

* k is Boltzmann's constant
k is wavenumber

energy density = (average energy) x (mode density)

- Rayleigh-Jeans: (does not agree with experiment)

$$\rho(\nu) = n(\nu) \times \mathrm{k}T = \frac{8\pi n_r^3 \nu^2}{c^3}\mathrm{k}T$$

$\rho \to \infty$ as ν increases ("ultraviolet catastrophe")

- Plank: (1918-Nobel Prize)

$E_n = nh\nu$ light energy is quantized!

$$\langle \mathrm{E} \rangle = \frac{\Sigma \mathrm{E}_n \mathrm{P}_n}{\Sigma P_n} = \frac{\sum_{n=0}^{\infty} nh\nu e^{\frac{-nh\nu}{\mathrm{k}T}}}{\sum_{n=0}^{\infty} e^{\frac{-nh\nu}{\mathrm{k}T}}} = h\nu \frac{\sum_{n=0}^{\infty} n\left(e^{\frac{-h\nu}{\mathrm{k}T}}\right)^n}{\sum_{n=0}^{\infty}\left(e^{\frac{-h\nu}{\mathrm{k}T}}\right)^n}$$

remember:

$$\sum_{n=0}^{\infty} n x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2}$$

so if: $x = e^{\frac{-h\nu}{kT}}$

$$\langle E \rangle = h\nu \frac{e^{\frac{-h\nu}{kT}}}{\left(1-e^{\frac{-h\nu}{kT}}\right)^2} \times \frac{1}{\sum_{n=0}^{\infty}\left(e^{\frac{-h\nu}{kT}}\right)^n} = \frac{h\nu\, e^{\frac{-h\nu}{kT}}}{\left(1-e^{\frac{-h\nu}{kT}}\right)^2} \times \left(1-e^{\frac{-h\nu}{kT}}\right) = \frac{h\nu\, e^{\frac{-h\nu}{kT}}}{1-e^{\frac{-h\nu}{kT}}}$$

$$\frac{1}{\sum_{n=0}^{\infty} x^n} = \frac{1}{\frac{1}{1-x}} = 1-x$$

average energy

$$\langle E \rangle = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} \quad \text{(Planck)}$$

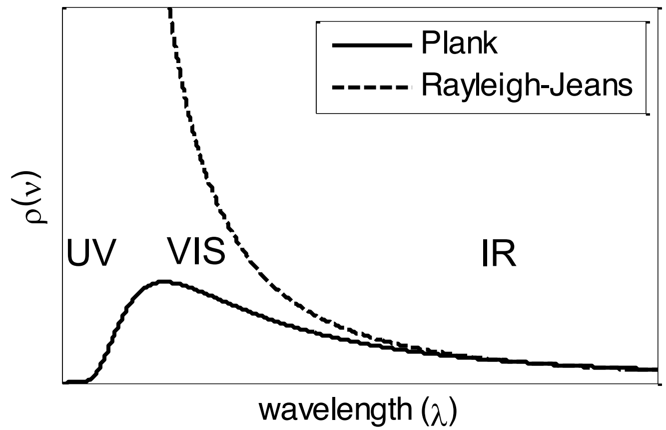
for $h\nu << kT$, $\langle E \rangle \approx KT$ (Boltzmann)

energy density

$$\rho(\nu)d\nu = \frac{8\pi\nu^2 n_r^3}{c^3} \frac{h\nu\, d\nu}{e^{\frac{h\nu}{kT}} - 1} \quad \text{(Planck)}$$

$$\rho(\nu) = \underbrace{n_\nu(\nu)}_{\text{mode density}} \times \underbrace{h\nu}_{\text{quantized energy}} \times \underbrace{\frac{1}{e^{\frac{h\nu}{kT}} - 1}}_{\text{\# of quanta in a cavity mode } N_p}$$

energy density = (average energy) x (mode density)



average # of photons / mode

$$\langle N_p \rangle = \frac{\langle E \rangle}{h\nu} = \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

for $h\nu << kT$

$$\langle N_p \rangle \simeq \frac{kT}{h\nu}$$ (Rayleigh-Jeans limit)

$h\nu \sim 1\ eV$ in near IR

$1\ eV = 2.4 \times 10^{14}\, Hz$

$\mathrm{k}T = \frac{1}{40}\ eV$ at rooom temp.

$\lambda = \frac{c}{\nu} = 1.25\ \mu m$

$\langle N_p \rangle \sim e^{-40} = 4.2 \times 10^{-18}$

Find spacing between modes in a 5m^3 room:

$$\frac{\#\text{ of modes}}{\text{frequency}} = n_\nu(\nu)\ V = \frac{8\pi\nu^2}{c^3} V\ ;\ 7 \times 10^6 \frac{\text{modes}}{Hz}$$

[Recall: $n_\nu(\nu)$ = (# of modes/frequency/volume)]