# Laser Oscillation: a different viewpoint

Before stimulated emission becomes important, we must obtain the initial photon from the "noise".

The laser oscillation builds up from the spontaneous emission "noise" emitted from the upper state until the coherent photon flux saturates the gain.

(1) $A_{21}N\,V$ - rate of generation of spontaneous photons into all frequencies. Modes are separated by: $\Delta\nu = c/(2nd)$

(2) $g(\nu)\Delta\nu = g(\nu)\left[c/(2nd)\right]$ - fraction of emission that appears in the interval $\Delta\nu = c/(2nd)$

(3) only the generated $TEM_{0,0,q}$ has a high Q, but there are $\dfrac{8\pi n^3\nu^2}{c^3}V \times \Delta\nu$ - modes in the volume

The rate of increase of photons in the $TEM_{0,0,q}$ mode caused by spontaneous emission:

$$\left.\frac{dN_p}{dt}\right|_{spont.} = \left(A_{21}N_2V\right)\left[g(\nu)\frac{c}{2nd}\right]\frac{1\text{mode}}{\#\text{modes} = \left(\dfrac{8\pi n^2\nu^2}{c^2}\right)\dfrac{c}{2nd}\times V}$$

$$\left.\frac{dN_p}{dt}\right|_{spont.} = N_2c\left[A_{21}\frac{\lambda^2}{8\pi}g(\nu)\right] = N_2c\sigma_{SE} \longleftarrow \text{Stimulated emission cross-section}$$

Photons in the cavity mode bounce back and forth between the 2 mirrors, being amplified by G per pass, with some of them escaping by mirror transmission.

If $N_p$ is starting # of photons, then $GR_1GR_2N_p$ returns after a round trip taking $2nd/c$ seconds.

(i.e. $\Delta N_p = G^2R_1R_2N_p - N_p) \Rightarrow$

$$\left.\frac{dN_p}{dt}\right|_{\text{cavity with gain}} = \frac{G^2R_1R_2 - 1}{2nd/c}N_p$$

<u>Gain + spontaneous emission:</u>

$$\frac{dN_p}{dt} = \underbrace{\frac{G^2R_1R_2 - 1}{2nd/c}N_p}_{\text{takes over at large } N_p} + \underbrace{N_2c\sigma_{SE}}_{\text{"works" in the beginning at small } N_p} \qquad (1)$$

If $N_p \sim 0$, $N_p$ still increases but the output (caused by spontaneous emission) is extremely small:

For $\sigma = 3\times10^{-12}\,cm^2$ (large!!)

$N_2 = 10^{12}\,cm^{-3}$ (typical) $\Rightarrow N_2c\sigma_{SE} = 9\times10^{10}\,s^{-1}$

if $h\nu = 1\ eV$ $(1.6\times10^{-19}J)$, then $P = 14.4\times10^{-9}W$ (negligible)

If $N_p$ is large enough, # photons grows exponentially:

$$N_p(t) = N_p(0)\exp\left[\frac{G^2R_1R_2 - 1}{2nd/c}t\right]$$

Then, saturation becomes important and eventually 1st term becomes negative (but small) and exactly balances the positive 2nd term so that we have a steady-state laser.

In the steady-state:

$$\frac{P_{out}}{h\nu} = \frac{N_p}{(2nd/c)} \underbrace{(1 - R_1 R_2)}_{\text{survived photons after a roundtrip}}$$

$$\frac{dN_p}{dt} = 0 \Rightarrow \text{see (1)} \rightarrow \left(-G^2 R_1 R_2 + 1\right) N_p = \frac{2nd}{c} N_2^{(S)} c\sigma_{SE}$$

$$\left(1 - G^2 R_1 R_2\right) \frac{P_{out}}{h\nu} \frac{1}{1 - R_1 R_2} = N_2 c\sigma_{SE}$$

$(S) \equiv saturation$

For typical values: $P_{out} = 10\ mW$ $\quad R_1 = 1$ $\quad R_L = 0.9$

$h\nu = 1\ eV$ $\quad N_2^{(S)} = 10^{12}\, cm^{-3}$ $\quad \sigma_{SE} = 3 \times 10^{-12}\, cm^2$

$$\boxed{\left(1 - G^2 R_1 R_2\right) = \frac{h\nu}{P_{out}} \left(1 - R_1 R_2\right) N_2^{(S)} c\sigma_{SE}} \left(\sim 10^{-7}\right) \qquad (2)$$

For any computational purpose $G^2 = \frac{1}{R_1 R_2}$

The fact that the gain does saturate at a value slightly less than the loss, implies that the laser will have a finite, nonzero, spectral width caused by the "noise" contributed by the spontaneous emission. (minimum laser linewidth, as follows…)

# Minimum Laser Linewidth

For Fabry-Perot cavity (see 6.3.3 in the text):

$$I = I_0 \frac{1}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\sin^2\theta} \qquad \theta = \frac{2\pi n d\nu}{c} \qquad (3)$$

For cavity with gain, using same approach as in 6.3.1-6.3.3, we obtain $R_1R_2 \rightarrow G^2R_1R_2$

$$P(\nu) = \frac{K}{\left[1-G_S\left(R_1R_2\right)^{1/2}\right]^2 + 4G_S\left(R_1R_2\right)^{1/2}\sin^2\left[\frac{2\pi d}{c}(\nu-\nu_q)\right]}$$

$K$ ← constant

at $\nu = \nu_q, P(\nu)$ has a peak (same as $I$ in (3) has a peak at $\theta = a\pi$ )

For FWHM: $$4G_S\left(R_1R_2\right)^{1/2}\left[\frac{2\pi d}{c}\left(\frac{\Delta\nu_{osc}}{2}\right)\right]^2 = \left[1-G_S(R_1R_2)^{1/2}\right]^2$$

$$\Delta\nu_{osc} = \frac{1-G_S(R_1R_2)^{1/2}}{\pi\left[G_S\left(R_1R_2\right)^{1/2}\right]^{1/2}}\frac{c}{2d}$$

$\Delta\nu$ - very small

$$G_S(R_1R_2) \simeq 1 \Rightarrow$$

$$1-G_S^2(R_1R_2) = \left[1-G_S(R_1R_2)^{1/2}\right]\left[1+\underbrace{G_S(R_1R_2)^{1/2}}_{2}\right] = 2\left[1-G_S(R_1R_2)^{1/2}\right]$$

$$\Delta\nu_{osc} = \frac{1-G^2{}_S(R_1R_2)}{2\pi\left[G_S\left(R_1R_2\right)^{1/2}\right]}\frac{c}{2d} = \frac{c}{4\pi d}\left[1-G_S{}^2(R_1R_2)\right] \qquad (*)$$

use $1 - G_S(R_1R_2) = \frac{h\nu}{P_{out}}\left(1 - R_1R_2\right) N_2^{(S)} c\sigma_{SE}$ (from (2)) (**)

photon lifetime ⟶ $\tau_p = \frac{\tau_{RT}}{1 - R_1R_2} = \frac{2\, d/c}{1 - R_1R_2}$

$$\Delta\nu_{1/2} = \frac{1}{2\pi\tau_p} = \frac{c}{2\pi d}\left(1 - R_1R_2\right)$$ - see 6.4.5 (4)

$$\Delta\nu_{osc} = \frac{h\nu}{P_{out}} \Delta\nu_{1/2} N_2^{(S)} c\sigma_{SE}$$ (see (4), (*) and (**)) (5)

gain coefficient $\left(N_2^{(S)} - \frac{g_2}{g_1} N_1^{(S)}\right)\sigma = \text{loss}/\text{unit length}$

$$N_2^{(S)}\sigma_{SE}\left(1 - \frac{g_2 N_1^{(S)}}{g_1 N_2^{(S)}}\right) = \frac{1}{2d}\ln\frac{1}{R_1R_2} = \frac{1}{2d}\ln\left[\frac{1}{1-\underbrace{\left(1 - R_1R_2\right)}}\right] \simeq \frac{1 - R_1R_2}{2d}$$

$\ln\left[1 + \left(1 - R_1R_2\right)\right] \approx 1 - R_1R_2$

$$N_2^{(S)}\sigma_{SE} \simeq \frac{1 - R_1R_2}{2d}\left(1 - \frac{g_2 N_1^{(S)}}{g_1 N_2^{(S)}}\right)^{-1}$$ (6)

Using (4),(5),(6)

$$\boxed{\Delta\nu_{osc} = 2\pi \frac{h\nu}{P_{out}}\left(1 - \frac{g_2 N_1^{(S)}}{g_1 N_2^{(S)}}\right)^{-1}\left(\Delta\nu_{1/2}\right)^{1/2}}$$ - in cavity with gain!

Schawlow – Townes formula

use:

$$N_1 = 0 \qquad \text{(ideal system)}$$

$$\nu = 2.4\times 10^{14}\ Hz$$

$$Q = 5\times 10^{7}$$

$$P = 10\ mW$$

$$\Rightarrow \Delta\nu_{1/2} = \frac{\nu}{Q} = 5\ MHz \qquad \text{(for passive cavity)}$$

$$\Rightarrow \Delta\nu_{osc} \sim 10^{-3}\,Hz$$

Even for $N_1 = 0$ (which is normally the case) $\Delta\nu_{osc}$ is very small. (minimum laser linewidth)

In practice, perturbations in the mirror separation completely overwhelm the foregoing limit.

A "$\delta$ -function" for the spectral representation of the laser is an excellent approximation.