Solution to Exam 2

1. $[D] = [U] \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} [U^+]$

$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\qquad \det(A - \lambda I) = 0$

$\Rightarrow \quad \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$

$\underline{\lambda = 1}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ normalized

$(A - \lambda I)\, x = 0 \quad$ gives $\quad x = \begin{bmatrix} -i \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\underline{\lambda = -1}$

$(A - \lambda I)\, x = 0 \quad$ gives $\quad x = \begin{bmatrix} i \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\Rightarrow \quad [U^+] = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$

$[U] = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$ $\qquad \Rightarrow \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2. Hamiltonian of the system

$$H = \begin{bmatrix} \varepsilon & t & 0 & 0 & 0 & 0 \\ t & \varepsilon & t & 0 & 0 & 0 \\ 0 & t & \varepsilon & t & 0 & 0 \\ 0 & 0 & t & \varepsilon & t & 0 \\ 0 & 0 & 0 & t & \varepsilon & t \\ 0 & 0 & 0 & 0 & t & \varepsilon \end{bmatrix}$$

$E\Psi = H\Psi$

$E\Psi_n = t\Psi_{n-1} + \Psi_n + t\Psi_{n+1}$

$\Psi_n = \Psi_0 e^{ikna} \quad \Rightarrow \quad E = \varepsilon + 2t\cos ka$

from periodic boundary condition

$ka = \frac{2\pi \nu}{6} = \frac{\pi \nu}{3} \qquad \nu = 0, 1, 2, \cdots 5$

$E = \varepsilon + 2t$

$\varepsilon + t$ (two fold degenerate)

$\varepsilon - t$ (two fold degenerate)

$\varepsilon - 2t$

3. $E^2 = BK^2$

circumferencial vector $c\vec{x}$

$\vec{K} \cdot c\vec{x} = 2\pi\nu$

$K_x = \left(\frac{2\pi\nu}{c}\right)$

$\Rightarrow E^2 = B\left[K_y^2 + \left(\frac{2\pi\nu}{c}\right)^2\right]$

$= BK^2 + E_{\nu 0}^2$ $\qquad E_{\nu 0} = \left(\frac{2\pi\nu}{c}\right)^2 B$

$K = \sqrt{(E^2 - E_{\nu 0}^2)/B}$

<u>for 1-D solid.</u>

$N(K) = \frac{L}{\pi} K$

$\Rightarrow N(E) = \frac{L}{\pi}\sqrt{\frac{E^2 - E_{\nu 0}^2}{B}}$

$D(E) = \frac{d\,N(E)}{dE}$

$= \sum_{\nu} \frac{L}{2\pi} \frac{2E}{\sqrt{E^2 - E_{\nu 0}^2}} \cdot \frac{1}{\sqrt{B}}$

$= \frac{L}{\pi\sqrt{B}} \sum_{\nu} \frac{E}{\left[E^2 - \left(\frac{2\pi\nu}{c}\right)^2 B\right]^{1/2}}$