1 Landauer's Formula

• In the previous section we introduced an expression for the tunneling current calculation of the form

$$J = -2e \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{dk_z}{2\pi} v_z T(E_z) \left[f(E_{FL}) - f(E_{FR}) \right] \tag{1}$$

To actually solve the problem and arrive at the Tsu-Esaki expression for the current we made a crucial assumption, that the distribution function is the Fermi - Dirac distribution function, valid only in thermal equilibrium (no applied bias).

• With current flow, the distribution function may be significantly perturbed from its equilibrium form. This was recognized early on by Landauer, who formulated the problem is a somewaht different way, by considering a one-dimensional system in which a constant current flow was forced to flow through a structure containing scatterers. The result in one dimension is the so-called LANDAUER FORMULA which will be derived here for a single channel case. It can be easily generalized to a multi-channel case.

• To get the Landauer's formula, we will consider a general barrier problem for a 1D conductor.

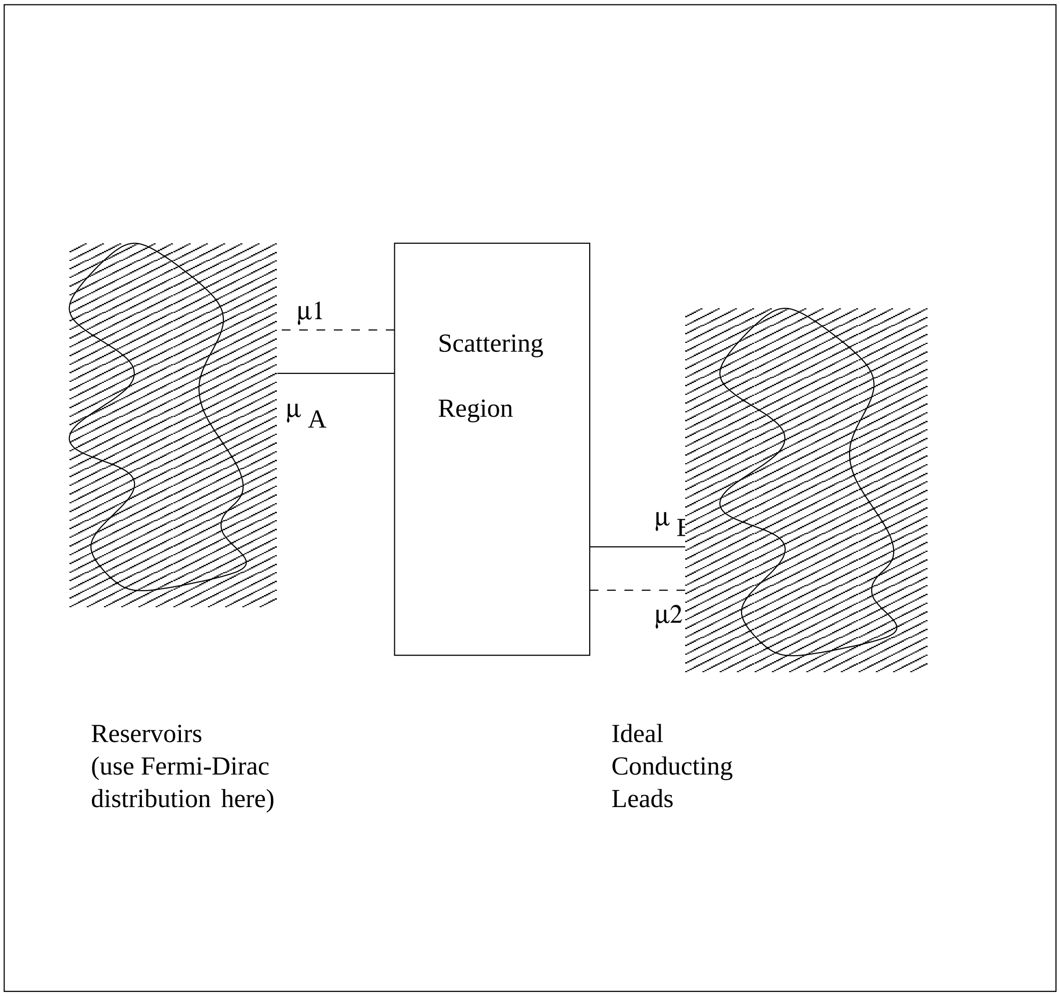

Figure 1: The Landauer formula

→ The IDEAL CONDUCTING LEADS connect the scattering region to the reservoirs that are characterized with chemical potentials μ_1 and μ_2 (quasi Fermi - Levels). These can be calculated from the corresponding electron densities.

→ The RESERVOIRS, or CONTACTS, randomize the phase of the injected and absorbed electrons through ineleastic processes, such as phonons, etc.

• For a 1D system, the current injected from the left and from the right is

$$J = 2e \int \frac{dk_z}{2\pi} v_z T(E_z) \left[f(\mu_1) - f(\mu_2) \right] \tag{2}$$

We can simplify this expression by using parabolic bands for which

$$E = \frac{\hbar^2 k^2}{2m} \tag{3}$$

which gives

$$\frac{dE}{dk} = \frac{\hbar^2}{m} k \tag{4}$$

or

$$dE = \frac{\hbar^2}{m} k dk \tag{5}$$

Therefore

$$\begin{aligned} J &= \frac{2e}{2\pi} \int dk_z \frac{m v_z}{m} T(E_z) \left[f(\mu_1) - f(\mu_2) \right] \\ &= \frac{2e}{2\pi} \int \frac{1}{m} \hbar k_z dk_z T(E_z) \left[f(\mu_1) - f(\mu_2) \right] \\ &= \frac{2e}{2\pi} \int \frac{\cancel{\hbar}}{\cancel{m}} \frac{\cancel{m}}{\hbar^{\cancel{2}}} dE_z T(E_z) \left[f(\mu_1) - f(\mu_2) \right] \\ &= \frac{2e}{h} \int dE_z T(E_z) \left[f(\mu_1) - f(\mu_2) \right] \end{aligned} \tag{6}$$

At $T = 0$, the distribution function in the reservoirs is of the form

$$f(\mu) = \theta(\mu - E) \quad = \quad 1, \qquad E \leq \mu \tag{7}$$

$$= \quad 0, \qquad \text{otherwise} \tag{8}$$

This imposes limits on the allowed energies, i.e.,

$$\begin{aligned} I &= \frac{2e}{\hbar} \left[\int_0^{\mu_1} T(E_z) dE_z - \int_0^{\mu_2} T(E_z) dE_z \right] \\ &= \frac{2e}{h} \int_{\mu_2}^{\mu_1} T(E_z) dE_z \end{aligned} \tag{9}$$

In the limit when the transmission coefficient is constant in the interval integration, which is usually the case in a linear - response regime, we have $T(E) = T \approx$ const., which gives

$$I = \frac{2e}{h} T(\mu_1 - \mu_2) \tag{10}$$

Now as a result of transmission and reflection about the barrier, there will be a reduction of the carrier density on one side and accumulation on the other side. Assuming that we are in a linear response regime, we can approximate this charge rearrangement by an average density in the leads characterized with different Fermi levels μ_A and μ_B. The voltage drop accross this structure is then

$$eV_a = \mu_A - \mu_B < \mu_1 - \mu_2 \tag{11}$$

• The next task is to relate μ_A and μ_B with μ_1 and μ_2 . To do so, we consider the following situation

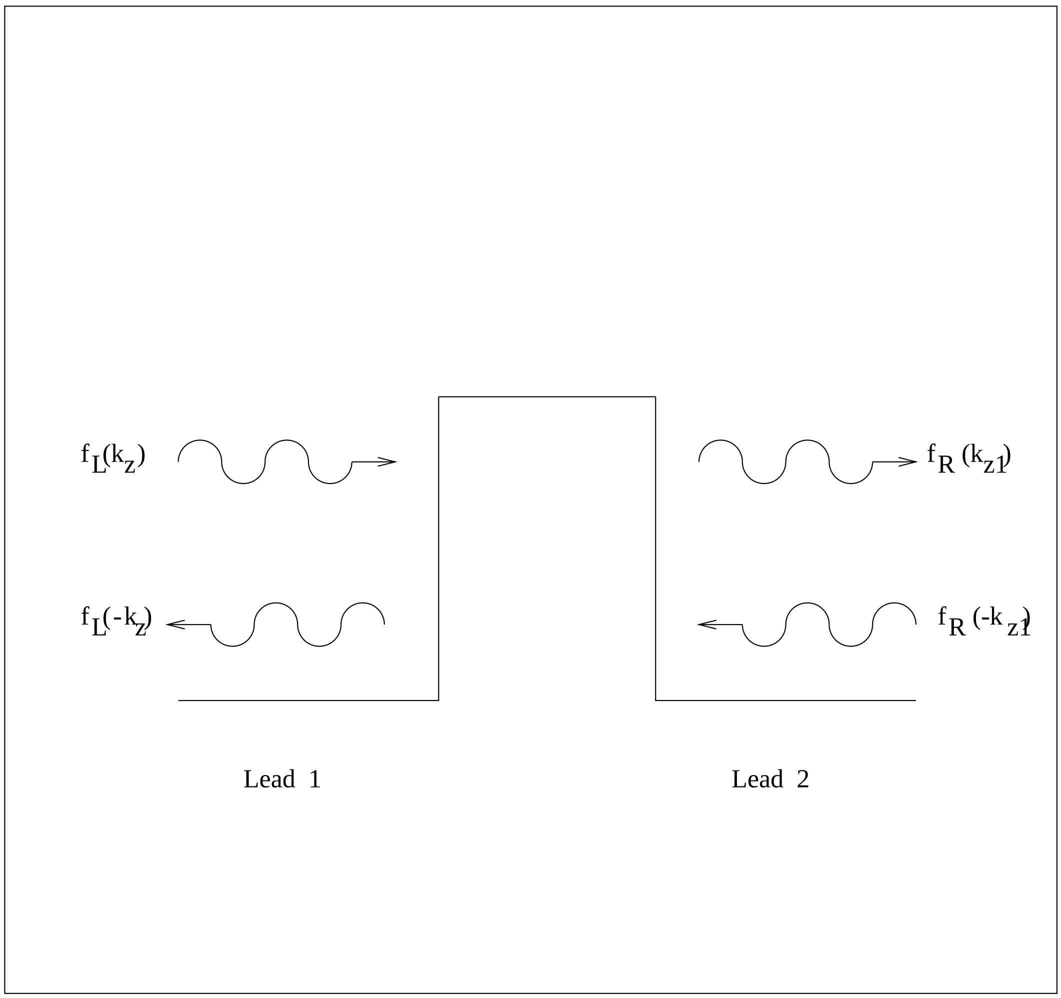

Figure 2: The Landauer formula

(a) The 1D density in the ideal lead 1 is given by

$$
\begin{aligned}
n_a &= \frac{2}{2\pi}\int_{-\infty}^{\infty} dk_z f_a(k_z) \\
&= \frac{1}{\pi}\int_{0}^{\infty} dk_z 2 f_a(k_z) \qquad (12)
\end{aligned}
$$

We can also express this density using the distribution function of the reservoirs, i.e.,

$$
n_a = \frac{1}{\pi}\int_0^{\infty} dk_z \left[f_L(k_z) + f_L(-k_z)\right] \qquad (13)
$$

where

$$
f_L(-k_z) = R f_L(k_z) + T f_R(-k_{z_1}) \qquad (14)
$$

Therefore

$$
\begin{aligned}
n_a &= \frac{1}{\pi}\int_0^{\infty} dk_z \left[f_L(k_z) + R f_L(k_z) + T f_R(-k_{z_1})\right] \\
&= \frac{1}{\pi}\int_0^{\infty} dk_z \left[(2-T) f_L(k_z) + T f_R(-k_{z_1})\right] \qquad (15)
\end{aligned}
$$

To summarize

$$
\frac{1}{\pi}\int_0^{\infty} dk_z 2 f_a(k_z) = \frac{1}{\pi}\int_0^{\infty} dk_z \left[(2-T) f_L(k_z) + T f_R(-k_{z_1})\right] \qquad (16)
$$

Similarly, for the electron density in lead 2, we have

$$
\begin{aligned}
n_b &= \frac{2}{2\pi}\int_{-\infty}^{\infty} dk_{z_1} 2 f_b(k_{z_1}) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk_{z_1} 2 f_b(k_{z_1}) \qquad (17) \\
&= \frac{1}{\pi}\int_0^{\infty} dk_{z_1} \left[f_R(k_{z_1}) + f_R(-k_{z_1})\right] \qquad (18)
\end{aligned}
$$

where

$$
f_R(k_{z_1}) = R f_R(-k_{z_1}) + T f_L(k_z) \qquad (19)
$$

Hence

$$
\begin{aligned}
n_b = \frac{1}{\pi}\int_0^{\infty} dk_{z_1} 2 f_b(k_{z_1}) &= \frac{1}{\pi}\int_0^{\infty} dk_{z_1} \left[f_R(-k_{z_1}) + R f_R(-k_{z_1}) + T f_L(k_z)\right] \\
&= \frac{1}{\pi}\int_0^{\infty} dk_{z_1} \left[(2-T) f_R(-k_{z_1}) + T f_L(k_z)\right] \qquad (20)
\end{aligned}
$$

• Subtracting the results obtained previously, and using the fact that $f(k_z) = f(\mu_1)$ and $f(-k_{z_1}) = f(\mu_2)$, we get

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{dk_z}{dE} 2\left[f(\mu_A) - f(\mu_B)\right] &= \frac{1}{\pi}\int_0^\infty \frac{dk_z}{dE}\left[(2-T)f(\mu_1)\right. \\ &+ \left. Tf(\mu_2) - (2-T)f(\mu_2) - Tf(\mu_1)\right] dE\end{aligned} \tag{21}$$

i.e.,

$$\begin{aligned}\int_0^\infty \frac{dk_z}{dE}\not{2}\left[f(\mu_A) - f(\mu_B)\right] dE &= \int_0^\infty \left[\not{2}(1-T)f(\mu_1)\right. \\ &- \left.\not{2}(1-T)f(\mu_2)\right] \frac{dk_z}{dE} dE\end{aligned} \tag{22}$$

Low T limit amd linear response

$$\begin{aligned}(\mu_A - \mu_B) &= (1-T)\mu_1 - (1-T)\mu_2 \\ &= (1-T)(\mu_1 - \mu_2)\end{aligned} \tag{23}$$

or

$$\mu_1 - \mu_2 = \frac{1}{1-T}(\mu_A - \mu_B) \tag{24}$$

• Substituting back into our original expression we have

$$\begin{aligned}I &= \frac{2e}{h}\frac{T}{1-T}(\mu_A - \mu_B) \\ &= \frac{2e}{h}\frac{T}{1-T} eV_a \\ &= \frac{2e^2}{h}\frac{T}{1-T} V_a\end{aligned} \tag{25}$$

The conductance is thus given by

$$G = \frac{2e^2}{h}\frac{T}{1-T} \tag{26}$$

• The result $(\mu_1 - \mu_2) = (\mu_A - \mu_B)/(1-T)$ suggests that the distribution of carriers in the leads is not a Fermi-Dirac function.

• The expression

$$G = \frac{2e^2}{h}\frac{T}{1-T} \tag{27}$$

is the single channel LANDAUER formula. The fundamental constant up-front $2e^2/h$ corresponds to a resistance of $R_c = \frac{h}{2e^2} = 12.907\ k\Omega$

• the factor $(1-T)$ in the denominator had resulted in a lot of confusion, but it actually boils down to how one performes the measurment

From the figure

$$\begin{aligned} V &= R_c I + V_a \qquad (28) \\ V &= \frac{\hbar^2}{2e^2}\frac{1}{T} I \qquad (29) \\ V_a &= \frac{h}{2e^2}\frac{1-T}{T} I \qquad (30) \end{aligned}$$

$$V = \frac{h}{2e^2}\frac{1}{T}\cancel{I} = R_c\cancel{I} + \frac{h}{2e^2}\left(\frac{1}{T} - 1\right)\cancel{I} \qquad (31)$$

$$\begin{aligned} \frac{h}{2e^2}\frac{1}{T} &= \frac{h}{2e^2}\frac{1}{T} - \frac{h}{2e^2} + R_c \qquad (32) \\ &\Rightarrow \boxed{R_c = \frac{h}{2e^2}} \qquad (33) \end{aligned}$$

↑ contact resistance

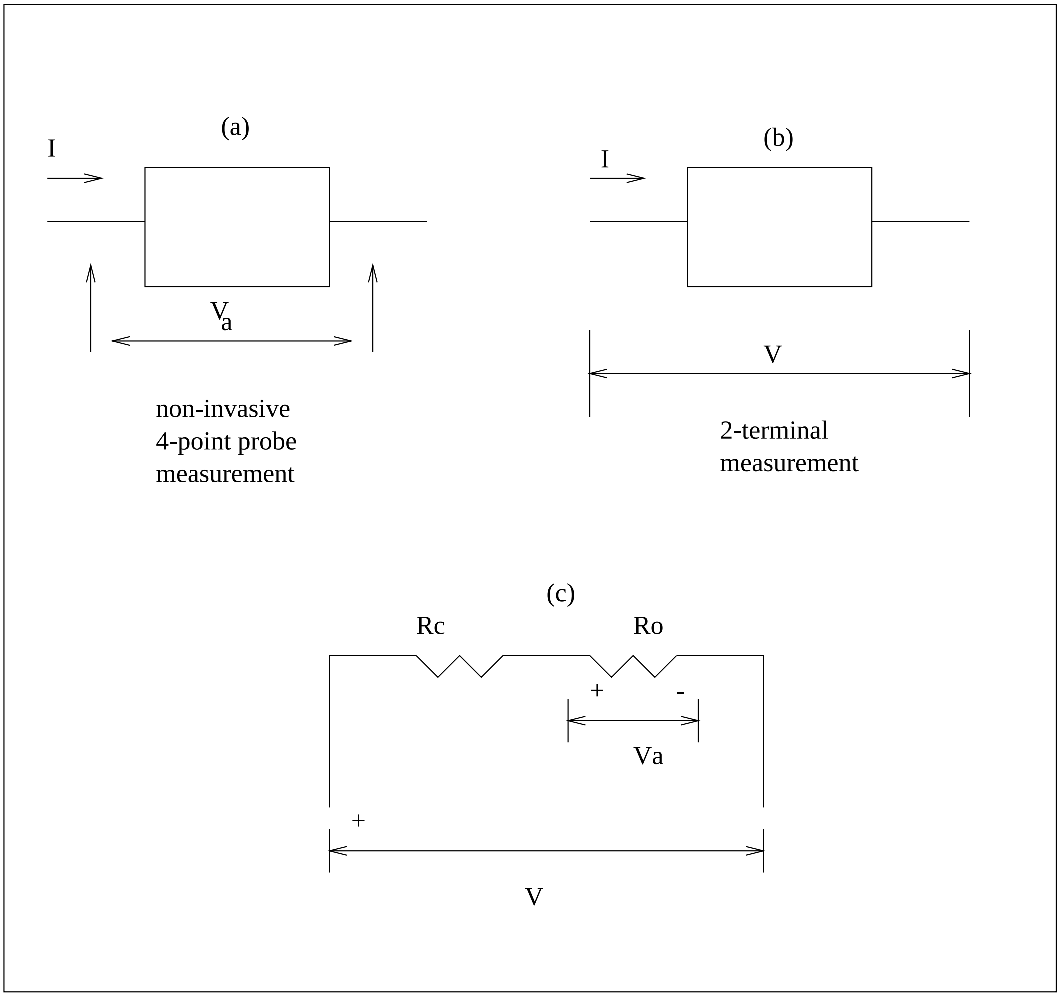

Figure 3: (a)$G = \frac{2e^2}{h}\frac{T}{1-T} = \frac{I}{V_a}$. (b) $G = \frac{I}{V} = \frac{2e^2}{h}T$. Here, in addition to measuring the voltage drop accross the scattering structure, one also measures the contact resistance. (c)Circuit equivalent of figure (3b), where $V = R_c I + V_a$