Harmonic Oscillator:

Motion in a Magnetic Field

* The Schrödinger equation in a magnetic field

$\Rightarrow$ The vector potential

* Quantized electron motion in a magnetic field

$\Rightarrow$ Landau levels

* The Shubnikov-de Haas effect

$\Rightarrow$ Landau-level degeneracy & depopulation

The Schrödinger Equation in a Magnetic Field

An important example of harmonic motion is provided by electrons that move under the influence of the LORENTZ FORCE generated by an applied MAGNETIC FIELD

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B} \qquad (16.1)$$

* From CLASSICAL physics we know that this force causes the electron to undergo CIRCULAR motion in the plane PERPENDICULAR to the direction of the magnetic field

* To develop a QUANTUM-MECHANICAL description of this problem we need to know how to include the magnetic field into the Schrödinger equation

⇒ In this regard we recall that according to FARADAY'S LAW a time-varying magnetic field gives rise to an associated ELECTRIC FIELD

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (16.2)$$

The Schrödinger Equation in a Magnetic Field

To simplify Equation 16.2 we define a VECTOR POTENTIAL A associated with the magnetic field

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \qquad (16.3)$$

* With this definition Equation 16.2 reduces to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A} \quad \therefore \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \qquad (16.4)$$

* Now the EQUATION OF MOTION for the electron can be written as

$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} \quad \therefore \quad \hbar\frac{\partial \mathbf{k}}{\partial t} = e\frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \quad \hbar\mathbf{k}(B) = \hbar\mathbf{k}_o + e\mathbf{A} \qquad (16.5)$$

1. MOMENTUM IN THE PRESENCE OF THE MAGNETIC FIELD

2. MOMENTUM PRIOR TO THE APPLICATION OF THE MAGNETIC FIELD

The Schrödinger Equation in a Magnetic Field

Inspection of Equation 16.5 suggests that in the presence of a magnetic field we REPLACE the momentum operator in the Schrödinger equation by

$$\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A} \qquad (16.6)$$

* To incorporate this result into the Schrödinger equation we recall that the first term on the LHS of its (B = 0) time-independent form represents the KINETIC ENERGY

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) + V\psi(x) = E\psi(x) \qquad (16.7)$$

NOTE THE THREE-DIMENSIONAL FORM

* Since kinetic energy is related to momentum as $p^2/2m$ this in turn suggests that we define the MOMENTUM OPERATOR as

$$\frac{\hbar}{i}\nabla = \frac{\hbar}{i}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right] \qquad (16.8)$$

The Schrödinger Equation in a Magnetic Field

With our definition of the momentum operator at ZERO magnetic field (Equation 16.8) we now use Equation 16.6 to obtain the momentum operator in the PRESENCE of a magnetic field

$$\left[\frac{\hbar}{i}\nabla + e\mathbf{A}\right] \qquad (16.9)$$

* With this definition we may now REWRITE the Schrödinger equation in a form that may be used to describe the motion of electrons in a magnetic field

$$\frac{1}{2m}\left[\frac{\hbar}{i}\nabla + e\mathbf{A}\right]^2 \psi(x) + V\psi(x) = E\psi(x) \qquad (16.10)$$

⇒ The first term in the brackets on the LHS of this equation is known as the CANONICAL momentum and is the momentum in the absence of a magnetic field

⇒ The entire term in brackets is called the MECHANICAL or KINEMATIC momentum and corresponds to the KINETIC ENERGY of the electron

Quantized Electron Motion in a Magnetic Field

We now apply the results of the preceding analysis to describe the motion of electrons in a magnetic field

* We assume that this magnetic field is CONSTANT and points in the *z*-direction

$$\mathbf{B} = B\hat{\mathbf{z}} \qquad (16.11)$$

* ONE possible choice of vector potential that satisfies this equation is known as the LANDAU GAUGE and is given by (you can check this using Equation 16.3)

$$\mathbf{A} = Bx\hat{\mathbf{y}} \qquad (16.12)$$

* With this choice of gauge the Schrödinger equation now becomes

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - i\frac{e\hbar Bx}{m}\frac{\partial}{\partial y} + \frac{(eBx)^2}{2m}\right]\psi = E\psi \qquad (16.13)$$

Quantized Electron Motion in a Magnetic Field

Equation 16.12 reveals that the magnetic field produces TWO effects

$$\left[-\frac{\hbar^2}{2m}\nabla^2 \underbrace{-i\frac{e\hbar Bx}{m}\frac{\partial}{\partial y}}_{1} + \underbrace{\frac{(eBx)^2}{2m}}_{2}\right]\psi = E\psi \qquad (16.13)$$

* The first effect is a derivative that COUPLES the motion in the *x*- and *y*-directions as we would EXPECT for a particle that undergoes CIRCULAR motion in the *xy*-plane

* The second effect is that the magnetic field generates a PARABOLIC MAGNETIC POTENTIAL of the form that we have studied for the harmonic oscillator!

* Now since the form of the vector potential we have chosen does NOT depend on *y* this suggests that we write the WAVEFUNCTION solutions for the electrons as

$$\psi(x,y) = u(x)e^{ik_y y} \qquad (16.14)$$

NOTE HOW THE y-COMPONENT CORRESPONDS TO A FREELY-MOVING PARTICLE

Quantized Electron Motion in a Magnetic Field

Substitution of our wavefunction into the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_c^2\left[x+\frac{\hbar k_y}{eB}\right]^2\right]u(x)=Eu(x) \qquad (16.15)$$

* This is just the Schrödinger equation for a one-dimensional HARMONIC OSCILLATOR with the magnetic-field dependent CYCLOTRON FREQUENCY

$$\omega_c=\frac{eB}{m} \qquad (16.16)$$

* An important difference with our previous analysis however is that the CENTER of the parabolic potential is NOT located at $x = 0$ but rather at

$$x_k=-\frac{\hbar k_y}{eB} \qquad (16.17)$$

Quantized Electron Motion in a Magnetic Field

Solution of the Schrödinger equation yields a quantized set of energy levels known as LANDAU LEVELS

$$E_n = \left[n + \frac{1}{2}\right]\hbar\omega_c \qquad (16.18)$$

* The wavefunctions are the usual HERMITE POLYNOMIALS and may be written as

$$\psi(x,y) = u(x)e^{ik_y y} = \sqrt{\frac{1}{2^n n!\sqrt{l_B}}}\exp\left[-\frac{(x-x_k)^2}{2l_B^2}\right]H_n\left[\frac{x-x_k}{l_B}\right]e^{ik_y y} \qquad (16.19)$$

$\Rightarrow$ where we have defined the MAGNETIC LENGTH l_B as

$$l_B = \sqrt{\frac{\hbar}{eB}} \qquad (16.20)$$

The Shubnikov-de Haas Effect

• An important property of the quantized Landau levels is that they are highly DEGENERATE

$$E_n = \left[n + \frac{1}{2} \right] \hbar \omega_c \qquad (16.18)$$

* By this we mean that each Landau level is able to hold a LARGE number of electrons

* To obtain an expression for this degeneracy we begin by assuming that the circular motion of the electrons occurs in a plane with dimensions $L_x \times L_y$

⇒ By assuming PERIODIC boundary conditions along the *y*-direction (ECE 352) we may write a quantization condition for the wavenumber k_y

$$k_y = \frac{2\pi}{L_y} j \, , \quad j = 0, 1, 2, \ldots \qquad (16.21)$$

WHEN A PARTICLE IS CONFINED IN A ONE-DIMENSIONAL BOX OF LENGTH Ly ITS ALLOWED WAVENUMBERS ARE QUANTIZED ACCORDING TO EQUATION 16.21

The Shubnikov-de Haas Effect

Since the CENTER COORDINATE of the harmonic oscillator must lie somewhere within the sample we may use Equation 16.21 to write the following condition

$$-L_x < x_k < 0 \quad \therefore \quad 0 < j < \frac{eBL_xL_y}{h} \qquad (16.22)$$

* According to Equation 16.22 EACH Landau level contains the same number of states at any given magnetic field

⇒ The number of states in each level PER UNIT AREA of the sample is just given by

$$\frac{j_{Max}}{L_xL_y} = \frac{eB}{h} \qquad (16.23)$$

⇒ Since each state within the Landau level can hold TWO electrons with OPPOSITE spins the number of ELECTRONS that can be held in each Landau level is

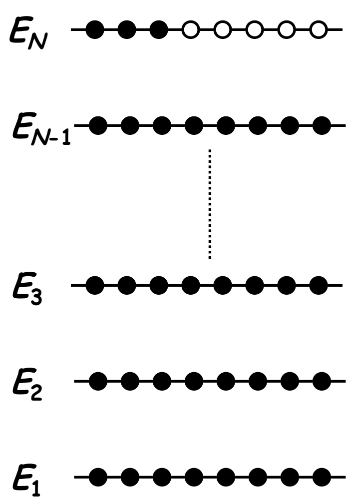
$$n = \frac{2eB}{h} \qquad (16.24)$$

The Shubnikov-de Haas Effect

Now let us consider what happens to a sample containing a FIXED number of electrons as we VARY the magnetic field

* Starting at some INITIAL magnetic field a specific number N of Landau levels will be occupied by electrons

$\Rightarrow$ $(N - 1)$ of these levels will be filled completely while the N^{th} will typically be PARTIALLY filled with the remaining electrons that cannot be accommodated in the lower levels

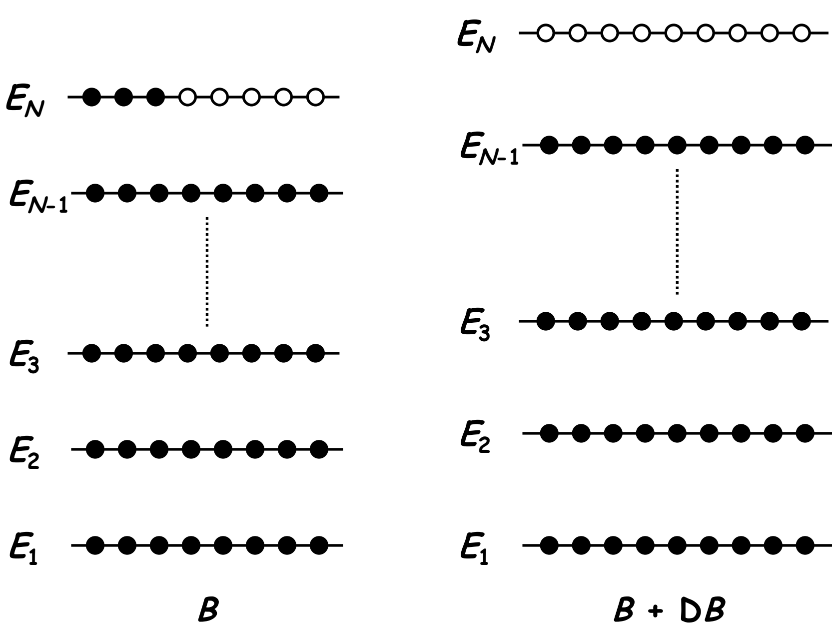


- *FILLING OF LANDAU LEVELS BY A FIXED NUMBER OF ELECTRONS AT AN ARBITRARY MAGNETIC FIELD*
- *EACH LANDAU LEVEL IS CAPABLE OF HOLDING THE SAME NUMBER OF ELECTRONS AND THESE LEVELS WILL BE FILLED IN A MANNER THAT MINIMIZES THE TOTAL ENERGY OF THE SYSTEM*
- *BECAUSE OF THIS AT ANY MAGNETIC FILLED THE LOWEST (N-1) LANDAU LEVELS WILL BE COMPLETELY FILLED BY ELECTRONS ACCOUNTING FOR (N-1)n ELECTRONS*
- *THE REMAINING ELECTRONS WILL BE ACCOMMODATED IN THE PARTIALLY-FILLED UPPERMOST LEVEL*

The Shubnikov-de Haas Effect

If we now raise the magnetic field we increase the ENERGY SPACING of the Landau levels and also increase their DEGENERACY

* Since more states are available in each level electrons DROP from higher levels to occupy empty states in the lower levels
* Consequently we eventually reach a point where the N^{th} Landau level is COMPLETELY emptied of electrons and the number of occupied Landau levels is now just N-1



- *EMPTYING OF THE UPPERMOST OCCUPIED LANDAU LEVEL IN AN INCREASING MAGNETIC FIELD*
- *WITH INCREASING MAGNETIC FIELD THE SPACING OF THE LANDAU LEVELS INCREASES BUT THE NUMBER OF ELECTRONS HELD BY EACH LEVEL ALSO INCREASES*
- *AS ELECTRONS DROP TO FILL NEW STATES THAT BECOME AVAILABLE WITH INCREASING FIELD THE UPPERMOST LANDAU LEVEL EVENTUALLY EMPTIES*
- *THIS PROCESS IS REFERRED TO AS MAGNETIC DEPOPULATION OF LEVELS*

The Shubnikov-de Haas Effect

The MAGNETIC DEPOPULATION we have described CONTINUES with increasing magnetic field until all electrons occupy only the LOWEST Landau level at VERY HIGH magnetic fields

* To obtain an expression for the magnetic field values at which the depopulations occur we consider a sample containing n_s electrons PER UNIT AREA

* Since n is the number of electrons per unit area that occupy EACH Landau level we require those values of the magnetic field for which the following ratio is an INTEGER

$$\frac{n_s}{n} = \frac{n_s h}{2eB} \equiv N_L \ , \quad N_L = 1, 2, 3, \ldots \qquad (16.25)$$

⇒ This relation shows that as expected the number of occupied Landau levels DECREASES with increasing magnetic field

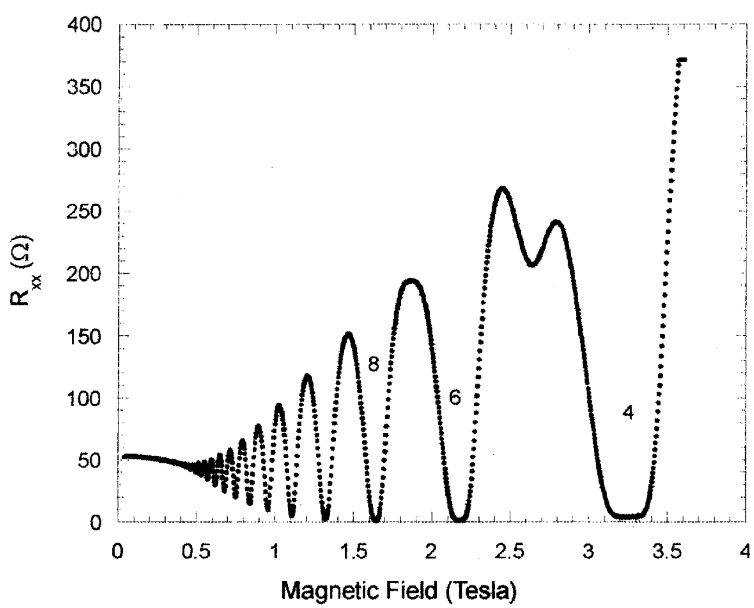
⇒ It also shows that an increasingly LARGER magnetic field increment is required to depopulate successively LOWER Landau levels

The Shubnikov-de Haas Effect

The depopulation of Landau levels can actually be seen in the MAGNETO-RESISTANCE of semiconductors which OSCILLATES at low temperatures

* The period of the oscillations INCREASES with magnetic field as expected from Equation 15.24 and the oscillations appear PERIODIC when plotted on an INVERSE-FIELD scale

⇒ The periodicity of these SHUBNIKOV-DE HAAS oscillations is often used in experiment as a means to determine the electron CARRIER DENSITY



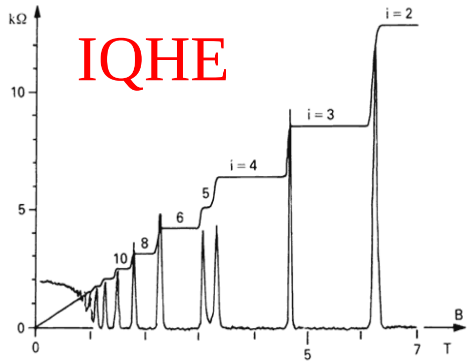
- *SHUBNIKOV-DE HAAS OSCILLATIONS MEASURED IN A GaAs/AlGaAs HETEROJUNCTION AT 4 K*

- *THE NUMBERS ON THE FIGURE INDICATE THE NUMBER OF OCCUPIED LANDAU LEVELS AT SPECIFIC VALUES OF THE MAGNETIC FIELD*

- *THE SPLITTING OF THE PEAK IN THE REGION OF AROUND 2.5 T RESULTS AS THE MAGNETIC FIELD BEGINS TO LIFT THE SPIN DEGENERACY OF THE ELECTRONS*

- *QUANTUM MECHANICS, D. K. FERRY, IOPP (2001)*

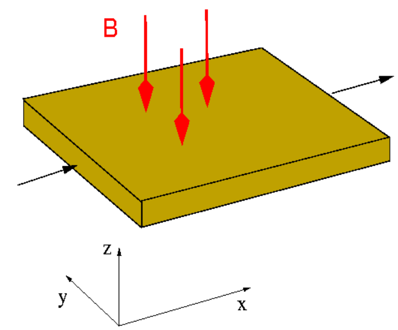
quantum Hall history

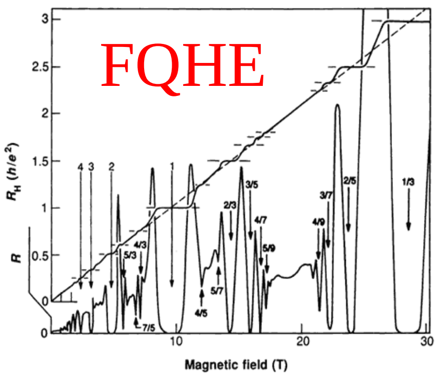


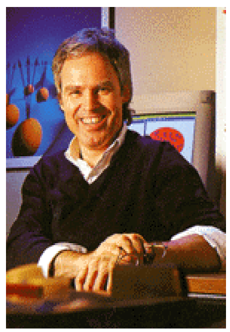
K. v. Klitzing

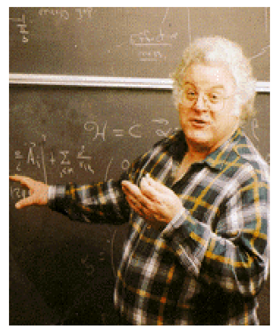
discovery: 1980

Nobel prize: 1985









D. Tsui **H. Störmer** **R. Laughlin**

discovery: 1982

Nobel prize: 1998