

Harmonic Oscillator:

Motion in a Magnetic Field

- * The Schrödinger equation in a magnetic field
 - ⇒ The vector potential
- * Quantized electron motion in a magnetic field
 - ⇒ Landau levels
- * The Shubnikov-de Haas effect
 - ⇒ Landau-level degeneracy & depopulation

The Schrödinger Equation in a Magnetic Field

An important example of harmonic motion is provided by electrons that move under the influence of the **LORENTZ FORCE** generated by an applied **MAGNETIC FIELD**

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B} \quad (16.1)$$

* From **CLASSICAL** physics we know that this force causes the electron to undergo **CIRCULAR** motion in the plane **PERPENDICULAR** to the direction of the magnetic field

* To develop a **QUANTUM-MECHANICAL** description of this problem we need to know how to include the magnetic field into the Schrödinger equation

⇒ In this regard we recall that according to **FARADAY'S LAW** a time-varying magnetic field gives rise to an associated **ELECTRIC FIELD**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (16.2)$$

The Schrödinger Equation in a Magnetic Field

To simplify Equation 16.2 we define a **VECTOR POTENTIAL** \mathbf{A} associated with the magnetic field

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \quad (16.3)$$

* With this definition Equation 16.2 reduces to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} \quad \therefore \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad (16.4)$$

* Now the **EQUATION OF MOTION** for the electron can be written as

$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} \quad \therefore \quad \hbar \frac{\partial \mathbf{k}}{\partial t} = e \frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \quad \boxed{\hbar \mathbf{k}(B)} = \boxed{\hbar \mathbf{k}_0} + e\mathbf{A} \quad (16.5)$$

1. MOMENTUM IN THE PRESENCE OF THE MAGNETIC FIELD

2. MOMENTUM PRIOR TO THE APPLICATION OF THE MAGNETIC FIELD

The Schrödinger Equation in a Magnetic Field

Inspection of Equation 16.5 suggests that in the presence of a magnetic field we **REPLACE** the momentum operator in the Schrödinger equation by

$$\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A} \quad (16.6)$$

* To incorporate this result into the Schrödinger equation we recall that the first term on the LHS of its ($B = 0$) time-independent form represents the **KINETIC ENERGY**

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V\psi(x) = E\psi(x) \quad (16.7)$$

NOTE THE THREE-DIMENSIONAL FORM

* Since kinetic energy is related to momentum as $p^2/2m$ this in turn suggests that we define the **MOMENTUM OPERATOR** as

$$\frac{\hbar}{i} \nabla = \frac{\hbar}{i} \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right] \quad (16.8)$$

The Schrödinger Equation in a Magnetic Field

With our definition of the momentum operator at **ZERO** magnetic field (Equation 16.8) we now use Equation 16.6 to obtain the momentum operator in the **PRESENCE** of a magnetic field

$$\left[\frac{\hbar}{i} \nabla + e\mathbf{A} \right] \quad (16.9)$$

* With this definition we may now **REWRITE** the Schrödinger equation in a form that may be used to describe the motion of electrons in a magnetic field

$$\frac{1}{2m} \left[\frac{\hbar}{i} \nabla + e\mathbf{A} \right]^2 \psi(x) + V\psi(x) = E\psi(x) \quad (16.10)$$

⇒ The first term in the brackets on the LHS of this equation is known as the **CANONICAL** momentum and is the momentum in the absence of a magnetic field

⇒ The entire term in brackets is called the **MECHANICAL** or **KINEMATIC** momentum and corresponds to the **KINETIC ENERGY** of the electron

Quantized Electron Motion in a Magnetic Field

We now apply the results of the preceding analysis to describe the motion of electrons in a magnetic field

* We assume that this magnetic field is **CONSTANT** and points in the z -direction

$$\mathbf{B} = B\hat{\mathbf{z}} \quad (16.11)$$

* **ONE** possible choice of vector potential that satisfies this equation is known as the **LANDAU GAUGE** and is given by (you can check this using Equation 16.3)

$$\mathbf{A} = Bx\hat{\mathbf{y}} \quad (16.12)$$

* With this choice of gauge the Schrödinger equation now becomes

$$\left[-\frac{\hbar^2}{2m} \nabla^2 - i \frac{e\hbar Bx}{m} \frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} \right] \psi = E\psi \quad (16.13)$$

Quantized Electron Motion in a Magnetic Field

Equation 16.12 reveals that the magnetic field produces **TWO** effects

$$\left[-\frac{\hbar^2}{2m} \nabla^2 \underset{1}{-i \frac{e\hbar Bx}{m} \frac{\partial}{\partial y}} + \underset{2}{\frac{(eBx)^2}{2m}} \right] \psi = E\psi \quad (16.13)$$

* The first effect is a derivative that **COUPLES** the motion in the x - and y -directions as we would **EXPECT** for a particle that undergoes **CIRCULAR** motion in the xy -plane

* The second effect is that the magnetic field generates a **PARABOLIC MAGNETIC POTENTIAL** of the form that we have studied for the harmonic oscillator!

* Now since the form of the vector potential we have chosen does **NOT** depend on y this suggests that we write the **WAVEFUNCTION** solutions for the electrons as

$$\psi(x, y) = u(x)e^{ik_y y} \quad (16.14)$$

NOTE HOW THE y -COMPONENT CORRESPONDS TO A FREELY-MOVING PARTICLE

Quantized Electron Motion in a Magnetic Field

Substitution of our wavefunction into the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega_c^2 \left[x + \frac{\hbar k_y}{eB} \right]^2 \right] u(x) = E u(x) \quad (16.15)$$

* This is just the Schrödinger equation for a one-dimensional **HARMONIC OSCILLATOR** with the magnetic-field dependent **CYCLOTRON FREQUENCY**

$$\omega_c = \frac{eB}{m} \quad (16.16)$$

* An important difference with our previous analysis however is that the **CENTER** of the parabolic potential is **NOT** located at $x = 0$ but rather at

$$x_k = -\frac{\hbar k_y}{eB} \quad (16.17)$$

Quantized Electron Motion in a Magnetic Field

Solution of the Schrödinger equation yields a quantized set of energy levels known as **LANDAU LEVELS**

$$E_n = \left[n + \frac{1}{2} \right] \hbar \omega_c \quad (16.18)$$

* The wavefunctions are the usual **HERMITE POLYNOMIALS** and may be written as

$$\psi(x, y) = u(x) e^{ik_y y} = \frac{1}{\sqrt{2^n n! \sqrt{l_B}}} \exp\left[-\frac{(x-x_k)^2}{2l_B^2}\right] H_n\left[\frac{x-x_k}{l_B}\right] e^{ik_y y} \quad (16.19)$$

⇒ where we have defined the **MAGNETIC LENGTH** l_B as

$$l_B = \sqrt{\frac{\hbar}{eB}} \quad (16.20)$$

The Shubnikov-de Haas Effect

- An important property of the quantized Landau levels is that they are highly **DEGENERATE**

$$E_n = \left[n + \frac{1}{2} \right] \hbar \omega_c \quad (16.18)$$

* By this we mean that each Landau level is able to hold a **LARGE** number of electrons

* To obtain an expression for this degeneracy we begin by assuming that the circular motion of the electrons occurs in a plane with dimensions $L_x \times L_y$

⇒ By assuming **PERIODIC** boundary conditions along the y -direction (ECE 352) we may write a quantization condition for the wavenumber k_y

$$k_y = \frac{2\pi}{L_y} j, \quad j = 0, 1, 2, \dots \quad (16.21)$$

WHEN A PARTICLE IS **CONFINED IN A ONE-DIMENSIONAL BOX OF LENGTH L_y ITS ALLOWED WAVENUMBERS ARE **QUANTIZED** ACCORDING TO EQUATION 16.21**

The Shubnikov-de Haas Effect

Since the **CENTER COORDINATE** of the harmonic oscillator must lie somewhere within the sample we may use Equation 16.21 to write the following condition

$$-L_x < x_k < 0 \quad \therefore \quad 0 < j < \frac{eBL_xL_y}{h} \quad (16.22)$$

* According to Equation 16.22 **EACH** Landau level contains the same number of states at any given magnetic field

⇒ The number of states in each level **PER UNIT AREA** of the sample is just given by

$$\frac{j_{Max}}{L_xL_y} = \frac{eB}{h} \quad (16.23)$$

⇒ Since each state within the Landau level can hold **TWO** electrons with **OPPOSITE** spins the number of **ELECTRONS** that can be held in each Landau level is

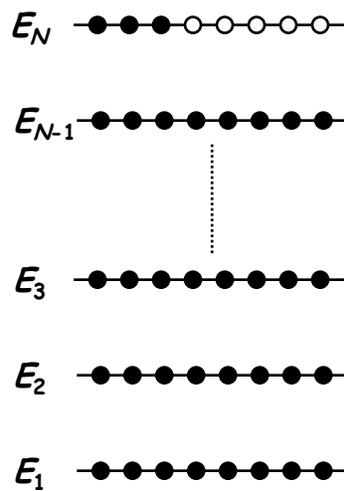
$$n = \frac{2eB}{h} \quad (16.24)$$

The Shubnikov-de Haas Effect

Now let us consider what happens to a sample containing a **FIXED** number of electrons as we **VARY** the magnetic field

* Starting at some **INITIAL** magnetic field a specific number N of Landau levels will be occupied by electrons

⇒ $(N - 1)$ of these levels will be filled completely while the N^{th} will typically be **PARTIALLY** filled with the remaining electrons that cannot be accommodated in the lower levels



• FILLING OF **LANDAU LEVELS** BY A **FIXED** NUMBER OF ELECTRONS AT AN ARBITRARY MAGNETIC FIELD

• EACH LANDAU LEVEL IS CAPABLE OF HOLDING THE **SAME** NUMBER OF ELECTRONS AND THESE LEVELS WILL BE FILLED IN A MANNER THAT **MINIMIZES** THE **TOTAL ENERGY** OF THE SYSTEM

• BECAUSE OF THIS AT ANY MAGNETIC FIELD THE LOWEST **(N-1)** LANDAU LEVELS WILL BE **COMPLETELY** FILLED BY ELECTRONS ACCOUNTING FOR $(N-1)n$ ELECTRONS

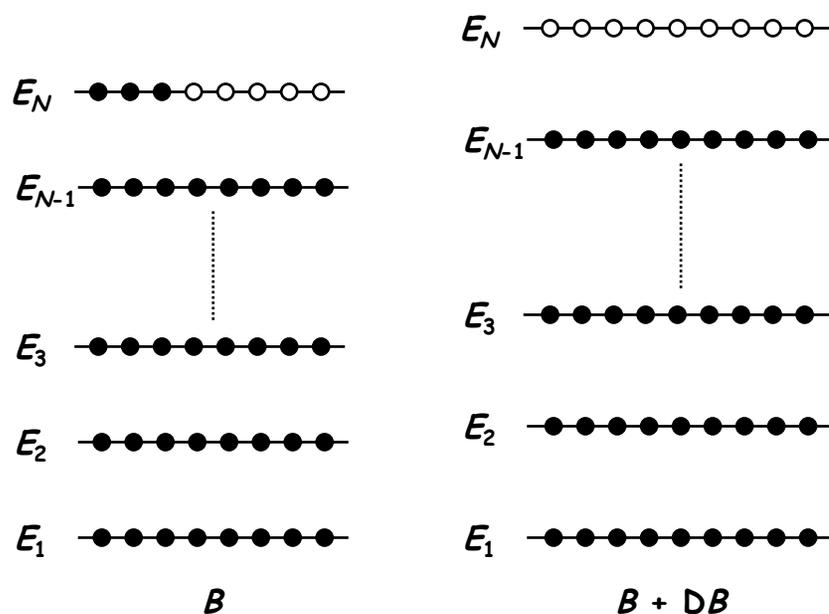
• THE REMAINING ELECTRONS WILL BE ACCOMMODATED IN THE **PARTIALLY-FILLED** UPPERMOST LEVEL

The Shubnikov-de Haas Effect

If we now raise the magnetic field we increase the **ENERGY SPACING** of the Landau levels and also increase their **DEGENERACY**

* Since more states are available in each level electrons **DROP** from higher levels to occupy empty states in the lower levels

* Consequently we eventually reach a point where the N^{th} Landau level is **COMPLETELY** emptied of electrons and the number of occupied Landau levels is now just $N-1$



• **EMPTYING OF THE UPPERMOST OCCUPIED LANDAU LEVEL IN AN INCREASING MAGNETIC FIELD**

• **WITH INCREASING MAGNETIC FIELD THE SPACING OF THE LANDAU LEVELS INCREASES BUT THE NUMBER OF ELECTRONS HELD BY EACH LEVEL ALSO INCREASES**

• **AS ELECTRONS DROP TO FILL NEW STATES THAT BECOME AVAILABLE WITH INCREASING FIELD THE UPPERMOST LANDAU LEVEL EVENTUALLY EMPTIES**

• **THIS PROCESS IS REFERRED TO AS MAGNETIC DEPOPULATION OF LEVELS**

The Shubnikov-de Haas Effect

The **MAGNETIC DEPOPULATION** we have described **CONTINUES** with increasing magnetic field until all electrons occupy only the **LOWEST** Landau level at **VERY HIGH** magnetic fields

* To obtain an expression for the magnetic field values at which the depopulations occur we consider a sample containing n_s electrons **PER UNIT AREA**

* Since n is the number of electrons per unit area that occupy **EACH** Landau level we require those values of the magnetic field for which the following ratio is an **INTEGER**

$$\frac{n_s}{n} = \frac{n_s h}{2eB} \equiv N_L, \quad N_L = 1, 2, 3, \dots \quad (16.25)$$

⇒ This relation shows that as expected the number of occupied Landau levels **DECREASES** with increasing magnetic field

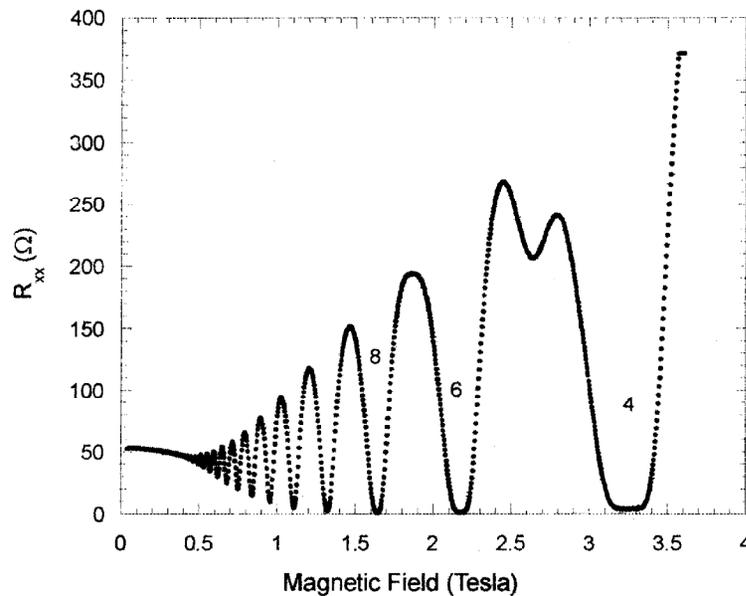
⇒ It also shows that an increasingly **LARGER** magnetic field increment is required to depopulate successively **LOWER** Landau levels

The Shubnikov-de Haas Effect

The depopulation of Landau levels can actually be seen in the **MAGNETO-RESISTANCE** of semiconductors which **OSCILLATES** at low temperatures

* The period of the oscillations **INCREASES** with magnetic field as expected from Equation 15.24 and the oscillations appear **PERIODIC** when plotted on an **INVERSE-FIELD** scale

⇒ The periodicity of these **SHUBNIKOV-DE HAAS** oscillations is often used in experiment as a means to determine the electron **CARRIER DENSITY**



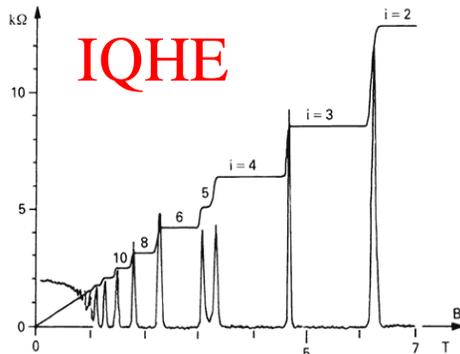
• **SHUBNIKOV-DE HAAS OSCILLATIONS** MEASURED IN A GaAs/AlGaAs HETEROJUNCTION AT 4 K

• THE NUMBERS ON THE FIGURE INDICATE THE NUMBER OF **OCCUPIED** LANDAU LEVELS AT SPECIFIC VALUES OF THE MAGNETIC FIELD

• THE **SPLITTING** OF THE PEAK IN THE REGION OF AROUND 2.5 T RESULTS AS THE MAGNETIC FIELD BEGINS TO LIFT THE **SPIN DEGENERACY** OF THE ELECTRONS

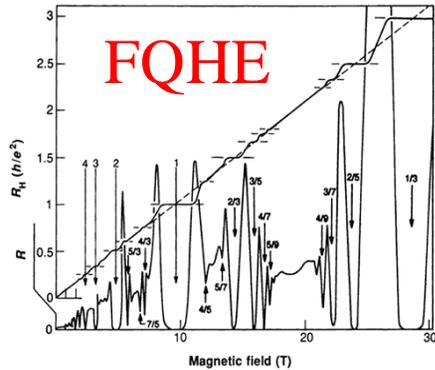
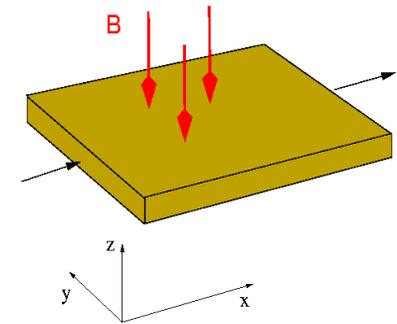
• QUANTUM MECHANICS, D. K. FERRY, IOPP (2001)

quantum Hall history



K. v. Klitzing

discovery: 1980
Nobel prize: 1985



D. Tsui H. Störmer R. Laughlin

discovery: 1982
Nobel prize: 1998