Practice Exam Solution

1. (a) $\sigma_{zz} = q^2 \frac{D}{WL} \frac{v^2 \tau}{d}$

$N = B k^d$

$D = \frac{dN}{dE} = B d\, k^{d-1} \underbrace{\frac{dk}{dE}}_{\frac{1}{\hbar v}}$

$= \frac{d \cdot N}{\hbar v k}$

$\sigma_{zz} = q^2 \frac{N}{WL} \frac{v^2 \tau}{\hbar v k}$

$= q^2 n_s \tau \left(\frac{v}{\hbar k} \right)$

$\equiv \frac{1}{m}$

(b) $\omega_c = \frac{qB\ 2\pi}{\hbar^2 \left(\frac{dA}{dE} \right)}$ $\quad A = \pi k_t^2 = \pi (k^2 - \text{constant})$

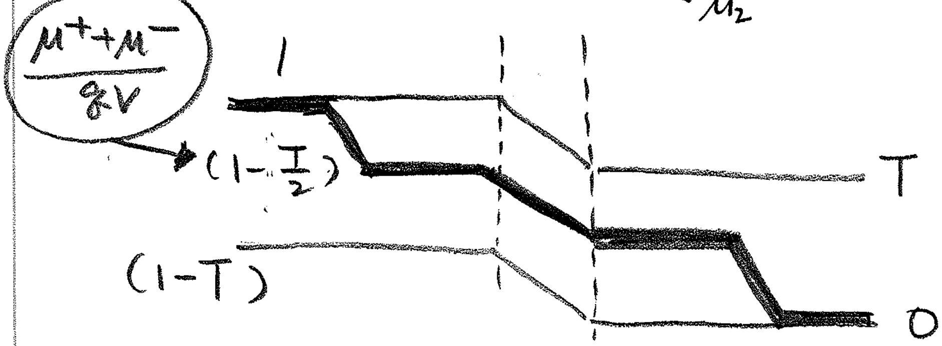
$= \frac{q v B}{\hbar k}$

$\therefore \frac{\omega_c \tau}{\sigma_{zz}} = \frac{B}{q n_s}$

2. (a), (c).

$$I_1^- = \frac{q}{h} M \cdot R \cdot \underbrace{qV}_{\leftarrow \mu_1^-}$$

$$I_2^- = \frac{q}{h} M \underbrace{T\, qV}_{\leftarrow \mu_2^-}$$



$$I_1^+ - I_1^- = I_2^- - I_2^+ = \frac{q^2}{h} M V \cdot T$$

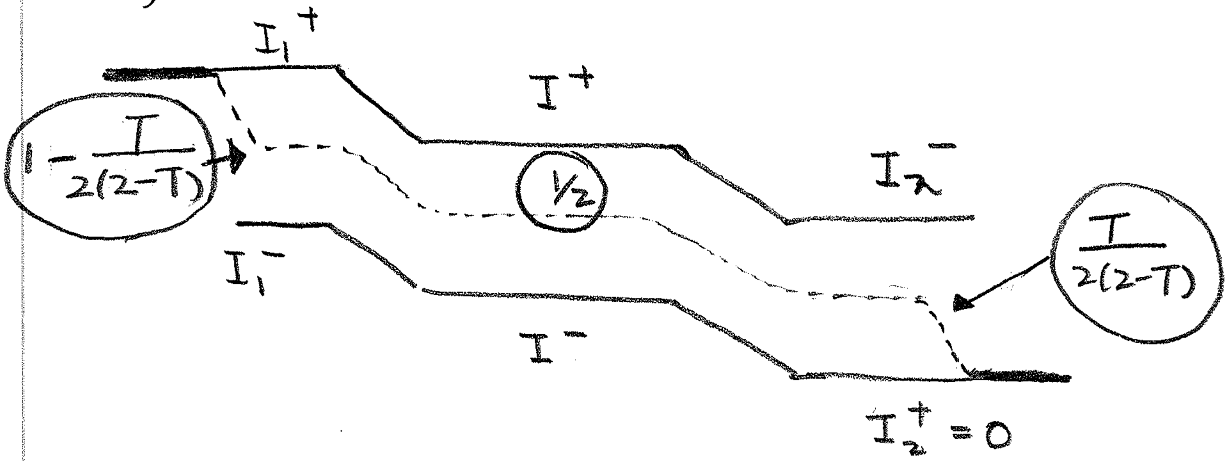
$$\text{Interface Resistance} = \frac{T/2 \cdot V}{\frac{q^2}{h} M V \cdot T} = \frac{1}{2} \frac{h}{q^2 M}$$

$$\text{Scatterer Resistance} = \frac{(1-T) \cdot V}{\frac{q^2}{h} M \cdot V \cdot T} = \frac{1-T}{T} \frac{h}{q^2 M}$$

(d) $$Y = \frac{q^2}{h} 2 \underbrace{[M - \bar{S}][M + \bar{S}]^{-1}[M]}_{M \begin{bmatrix} T & -T \\ -T & T \end{bmatrix} \begin{bmatrix} 1+R & T \\ T & 1+R \end{bmatrix}^{-1}}$$

$$= \frac{2MT}{(1+R)^2 - T^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1+R & -T \\ -T & 1+R \end{bmatrix} = \frac{MT}{(1-T)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b), (e)



$$I^+ = T I_1^+ + R I^- = T I_1^+ + R^2 I^+$$

$$I^- = R I^+$$

✓ $$I^+ = \frac{T}{1-R^2} I_1^+ = \frac{1}{2-T} I_1^+$$

✓ $$I^- = \frac{1-T}{2-T} I_1^+$$

✓ $$I_2^- = T I^+ = \frac{T}{2-T} I_1^+$$

✓ $$I_1^- = R I_1^+ + T I^- = I_1^+\left(1-T+\frac{T(1-T)}{2-T}\right)$$

$$= \frac{2(1-T)}{2-T} I_1^+$$

$$I = I_2^- - I_2^+ = I_1^+ - I_1^-$$

$$= I_1^+\left(1 - \frac{2(1-T)}{2-T}\right) = I_1^+ \frac{T}{2-T}$$

$$\frac{I_1^+ + I_1^-}{2} = I_1^+\left[1-\left(\frac{1}{2} - \frac{1-T}{2-T}\right)\right] = \left(1-\frac{T}{2(2-T)}\right) I_1^+$$

Interface Resistance

$$= \frac{\frac{T}{2(2-T)} V}{\frac{q^2}{h} M V \frac{T}{2-T}} = \frac{1}{2} \frac{q^2 M}{h}$$

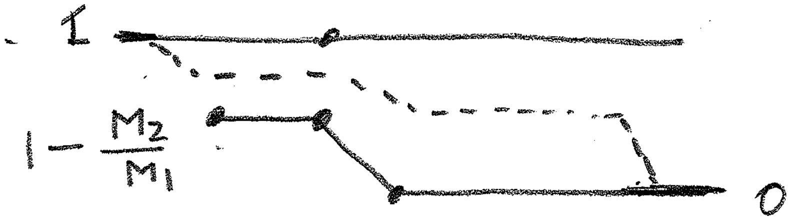
Scatterer Resistance

$$= \frac{V\left(\frac{1}{2} - \frac{T}{2(2-T)}\right)}{\frac{q^2}{h} M V \frac{T}{2-T}} = \frac{h}{q^2 M} \frac{2-2T}{2\cancel{(2-T)}} \frac{\cancel{2-T}}{T}$$

$$= \frac{h}{q^2 M} \frac{1-T}{T}$$

3. $$\begin{cases} I_1^- = \dfrac{q}{h} qV (M_1 - M_2) \\ \mu_1^- = qV \left(1 - \dfrac{M_2}{M_1}\right) \end{cases}$$

$$\begin{cases} I_2^- = \dfrac{q}{h} qV M_2 \\ \mu_2^- = qV = \mu_1^+ \end{cases}$$



Junction Resistance

$$\frac{\left(\frac{1}{2} - \frac{M_2}{2M_1}\right) V}{\frac{q^2}{h} M_2 V} = \frac{h}{q^2} \frac{1}{2} \left(\frac{1}{M_2} - \frac{1}{M_1}\right)$$

Interface 2

$$\frac{\frac{1}{2} V}{\frac{q^2}{h} M_2 V} = \frac{h}{q^2} \frac{1}{2M_2}$$

Interface 1

$$\frac{M_2/2M_1 \; V}{\frac{q^2}{h} M_2 V} = \frac{h}{q^2} \frac{1}{2M_1}$$

Total Resistance $= \dfrac{h}{q^2} \dfrac{1}{M_2}$

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$$E(k_x, k_y) = \frac{(\hbar k_x - qA_x)^2}{2m} + \frac{(\hbar k_y - qA_y)^2}{2m}.$$

5.

$$E'(k_x, k_y) = \varepsilon - t_x e^{i(k_x a - \phi_x)} - t_x e^{-i(k_x a - \phi_x)} - t_y e^{i(k_y a - \phi_y)} - t_y e^{-i(k_y a - \phi_y)}$$

$$= \varepsilon - 2t_x \cos(k_x a - \phi_x) - 2t_y \cos(k_y a - \phi_y)$$

$$\simeq (\varepsilon - 2t_x - 2t_y) + t_x(k_x a - \phi_x)^2 + t_y(k_y a - \phi_y)^2$$

$$\varepsilon = 2t_x + 2t_y, \quad t_x = t_y = \frac{\hbar^2}{2ma^2}$$

$$\phi_x = \frac{qA_x a}{\hbar}, \quad \phi_y = \frac{qA_y a}{\hbar}$$

6

(a) $\Sigma_1 = -\frac{i\Gamma_1}{2}$, $\Sigma_2 = -\frac{i\Gamma_2}{2}$.

$\Gamma_1 = \Gamma_1$, $\Gamma_2 = \Gamma_2$.

$$G^n = G\Sigma^{in}G^+$$
$$= G(\Gamma_1 f_1 + \Gamma_2 f_2)G^+$$
$$= G(\Gamma_1+\Gamma_2)G^+ \left(\frac{\Gamma_1 f_1 + \Gamma_2 f_2}{\Gamma_1+\Gamma_2}\right)$$
$$= A \cdot \left(\frac{\Gamma_1 f_1 + \Gamma_2 f_2}{\Gamma_1+\Gamma_2}\right)$$

$$I_1 = \frac{q}{h} \Gamma_1 A f_1 - \Gamma_1 G^n$$
$$= \frac{q}{h} \Gamma_1 A \left(f_1 - \frac{G^n}{A}\right)$$
$$= \frac{q}{h} \Gamma_1 A \left(f_1 - \frac{\Gamma_1 f_1 + \Gamma_2 f_2}{\Gamma_1+\Gamma_2}\right)$$
$$= \frac{q}{h} \frac{\Gamma_1 \Gamma_2}{\Gamma_1+\Gamma_2} A (f_1 - f_2)$$

(b).

$$\Gamma = \Gamma_1 + \Gamma_2 + D_0 A$$

$$\Gamma = \Gamma_1 + \Gamma_2 + D_0 A$$

$$\Sigma^{in} = \Gamma_1 f_1 + \Gamma_2 f_2 + D_0 G^n$$

$$G^n = G \Sigma^{in} G^+$$

$$= G(\Gamma_1 f_1 + \Gamma_2 f_2 + D_0 G^n) G^+$$

$$A = G \Gamma G^+$$

$$= G(\Gamma_1 + \Gamma_2 + D_0 A) G^+$$

$$\frac{G^n}{A} = \frac{\Gamma_1 f_1 + \Gamma_2 f_2 + D_0 G^n}{\Gamma_1 + \Gamma_2 + D_0 A} = \frac{\Gamma_1 f_1 + \Gamma_2 f_2}{\Gamma_1 + \Gamma_2}$$

$$I_1 = \frac{q}{h} \Gamma_1 A f_1 - \Gamma_1 G^n$$

$$= \frac{q}{h} \Gamma_1 A \left(f_1 - \frac{G^n}{A}\right)$$

$$= \frac{q}{h} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} A (f_1 - f_2)$$

P

7.

(a)

Treat "0" as device, rest as self energy

$$H = [2t_0]$$

$$\Sigma_1 = -t_0 e^{ika}, \quad \Sigma_2 = -t_0 e^{ika}$$

$$G = (E - 2t_0 + 2t_0 e^{ika})^{-1}$$

$$= \frac{1}{2t_0} \frac{1}{e^{ika} - \cos ka} = \frac{-i}{2t_0 \sin ka}$$

$$\frac{A}{2\pi} = \frac{i}{2\pi}\left(\frac{-2i}{2t_0 \sin ka}\right) = \frac{1}{2\pi t_0 \sin ka}$$

$$\cos ka = \frac{2t_0 - E}{2t_0}$$

$$\sin ka = \sqrt{1 - \left(\frac{2t_0 - E}{2t_0}\right)^2}$$

$$2t_0 \sin ka = \sqrt{(2t_0)^2 - (2t_0 - E)^2} = \sqrt{(4t_0 - E)E}$$

$$D(0,E) = \frac{A}{2\pi} = \frac{1}{\pi\sqrt{(4t_0 - E)E}}$$

(b) $\Gamma_1 = i[\Sigma_1 - \Sigma_1^+] = i[\Sigma_2 - \Sigma_2^+]$ (one level model).

$$\frac{G^n}{A} = \frac{\Gamma_1 f_1 + \Gamma_2 f_2}{\Gamma_1 + \Gamma_2} = \frac{f_1 + f_2}{2}$$

$$I_1 = \frac{q}{h}\, \Gamma_1 A f_1 - \Gamma_1 G^n$$

$$= \frac{q}{h}\, \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}\, A\, (f_1 - f_2)$$

$$= \frac{q}{h}\, \frac{\Gamma_1}{2}\, A\, (f_1 - f_2)$$

$$= \frac{q}{h}\, \frac{2 t_0 \sin ka}{2}\, \frac{1}{t_0 \sin ka}\, (f_1 - f_2)$$

$$= \frac{q}{h}\, (f_1 - f_2)$$

(c) Same as (a) but $\Sigma_2 = 0$.

$$G = \frac{1}{t_0} \frac{1}{e^{ika} - 2\cos ka} = -\frac{1}{t_0} e^{ika}.$$

$$\frac{A}{2\pi} = \frac{i}{2\pi}\left(-\frac{1}{t_0}\right) \cdot 2i\sin ka$$

$$= \frac{\sin ka}{\pi t_0} = \frac{\sqrt{(4t_0 - E)E}}{2\pi t_0^2}$$

8.

$$\begin{cases} M = \dfrac{2k}{2\pi/w} = \dfrac{kw}{\pi} \\ n_s = \dfrac{\pi k^2}{4\pi^2} \cdot 2 = \dfrac{k^2}{2\pi} \\ M = \dfrac{w}{\pi}\sqrt{2\pi n_s} = \sqrt{n_s}\; w\sqrt{\dfrac{2}{\pi}} \end{cases}$$

$$\begin{cases} M = \dfrac{\pi k^2}{4\pi^2/s} = \dfrac{k^2 s}{4\pi} \\ n = \dfrac{\frac{4}{3}\pi k^3}{8\pi^3} \cdot 2 = \dfrac{k^3}{3\pi^2} \\ M = \dfrac{s}{4\pi}(3\pi^2 n)^{2/3} = n^{2/3}\, s\, \dfrac{(3\pi^2)^{2/3}}{4\pi} \end{cases}$$