OPTICS: INTRODUCTION & FUNDAMENTALS

presented by

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University of Illinois at Urbana-Champaign

LECTURE OUTLINE

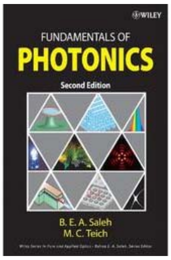
- Ray optics
- Wave optics
- Statistical optics
- Fourier optics
- Diffraction

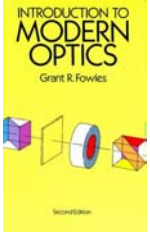
WHAT IS OPTICS?

- *Deals with the generation, propagation, and detection of light*

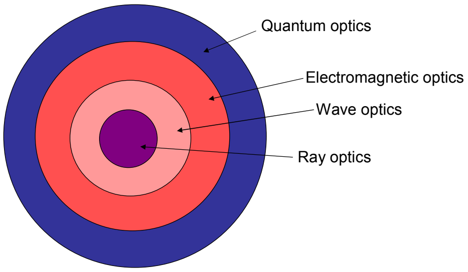
Optics ~ Photonics ~ Light

SOURCE MATERIAL





OPTICS THEORIES



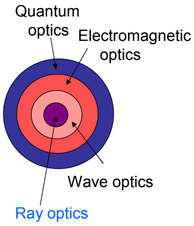
Quantum optics

Electromagnetic optics

Wave optics

Ray optics

OPTICS: THEORIES



Quantum optics

Electromagnetic optics

Wave optics

Ray optics

- *Ray optics*-Limit of wave optics when wavelength is infinitesimally small
- *Wave optics*-provides a description of optical phenomena using scalar wave theory
- *Electromagnetic optics*- provides most complete treatment of light within classical optics
- *Quantum optics*-provides a quantum mechanical description of the electromagnetic theory

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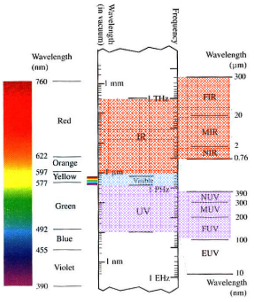
EM SPECTRUM



- Light is really, really fast!
- $c_o = 3.0 \times 10^8$ m/s

  = 30 cm/ns

  = 0.3 mm/ps

Image Source: Fundamentals of Photonics

INDEX OF REFRACTION

An optical medium is characterized by a quantity *n* >=1

Speed of light in vacuum

$$n = \frac{c_o}{c}$$

Speed of light in a medium

Optical pathlength

Travel time

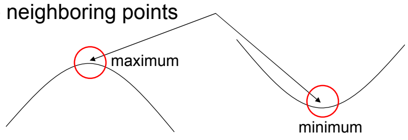
$$t = \frac{d}{c} = \frac{\circ{nd}}{c_o}$$

RAY OPTICS

- Geometrical optics
- Describes light as rays that travel in accordance with a set of geometrical rules
- Deals with location and direction of rays

FERMAT'S PRINCIPLE

- Rays traveling between two points follow a path such that the optical pathlength is an extremum (<u>point of inflection</u>) relative to neighboring points

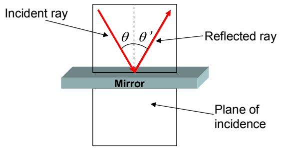


- Usually minimum, so
  Light rays travel along the path of least time

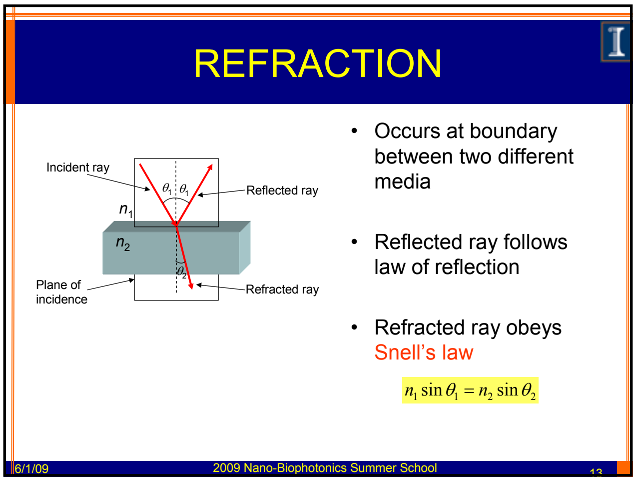
HERO'S PRINCIPLE

- *n* is the same everywhere in a <u>homogeneous</u> medium
- Path of min time = Path of min distance
- *Principle of path of min distance states that the path of min distance between points is a straight line*
- *Light rays travel in straight lines*

REFLECTION



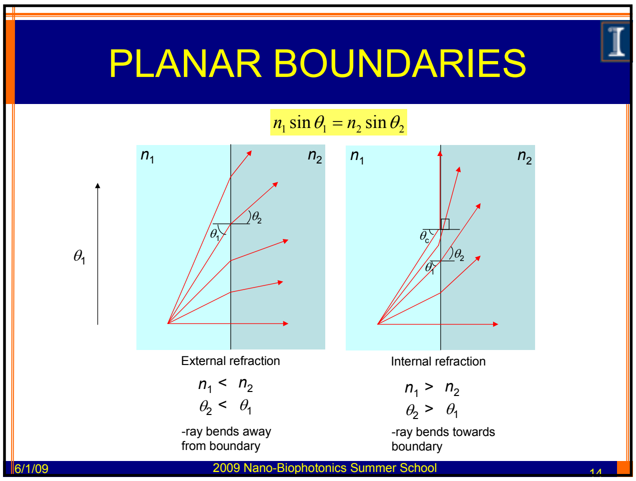
- Law of reflection:
  Reflected ray lies in plane of incidence



REFRACTION

- Occurs at boundary between two different media
- Reflected ray follows law of reflection
- Refracted ray obeys Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$



PLANAR BOUNDARIES

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

External refraction

$n_1 < n_2$

$\theta_2 < \theta_1$

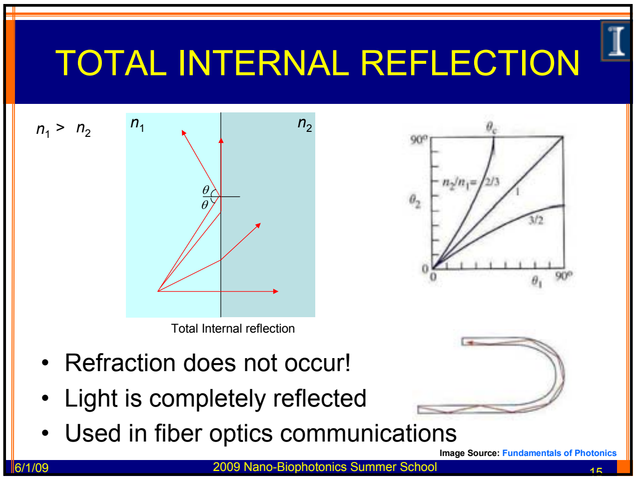
-ray bends away from boundary

Internal refraction

$n_1 > n_2$

$\theta_2 > \theta_1$

-ray bends towards boundary



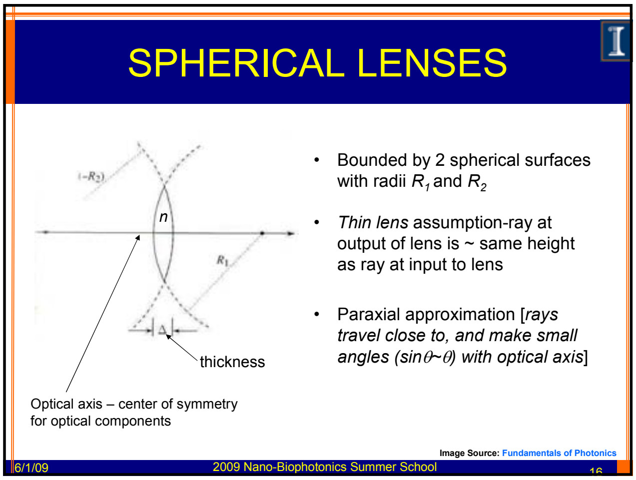
TOTAL INTERNAL REFLECTION

$n_1 > n_2$

Total Internal reflection

- Refraction does not occur!
- Light is completely reflected
- Used in fiber optics communications

Image Source: Fundamentals of Photonics

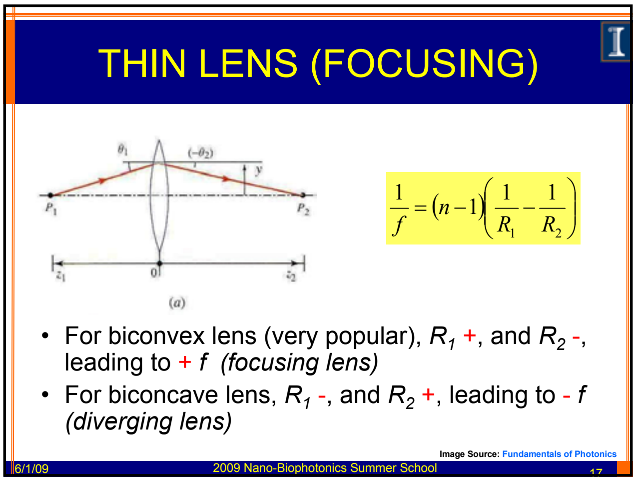


SPHERICAL LENSES

Optical axis – center of symmetry for optical components

- Bounded by 2 spherical surfaces with radii R_1 and R_2
- *Thin lens* assumption-ray at output of lens is ~ same height as ray at input to lens
- Paraxial approximation [*rays travel close to, and make small angles (sinθ~θ) with optical axis*]

Image Source: Fundamentals of Photonics

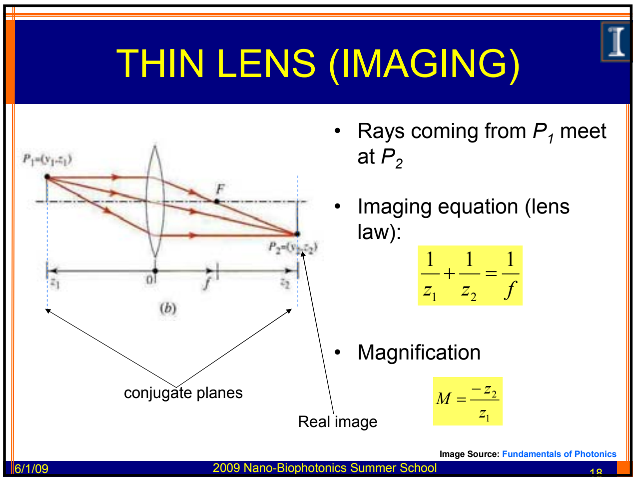


THIN LENS (FOCUSING)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

- For biconvex lens (very popular), R_1 +, and R_2 -, leading to + f *(focusing lens)*
- For biconcave lens, R_1 -, and R_2 +, leading to - f *(diverging lens)*

Image Source: Fundamentals of Photonics



THIN LENS (IMAGING)

conjugate planes

Real image

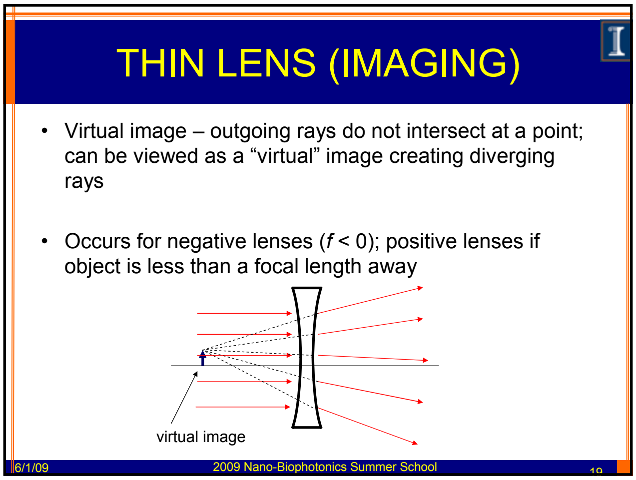
- Rays coming from P_1 meet at P_2
- Imaging equation (lens law):

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$$

- Magnification

$$M = \frac{-z_2}{z_1}$$

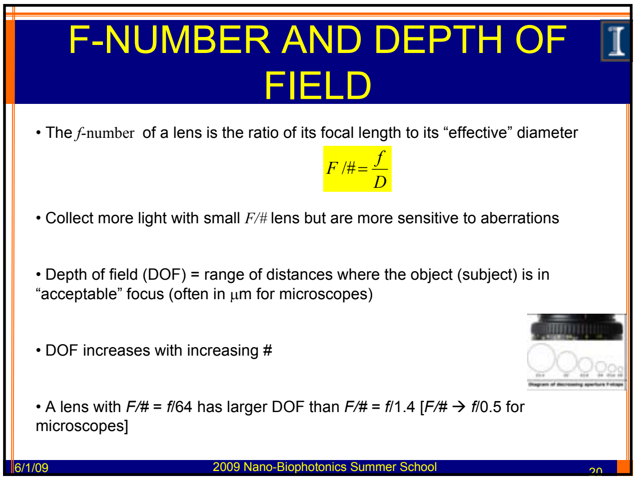
Image Source: Fundamentals of Photonics



THIN LENS (IMAGING)

- Virtual image – outgoing rays do not intersect at a point; can be viewed as a “virtual” image creating diverging rays
- Occurs for negative lenses (f < 0); positive lenses if object is less than a focal length away

virtual image



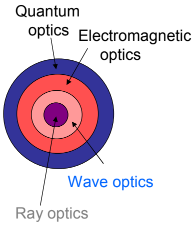
F-NUMBER AND DEPTH OF FIELD

- The f-number of a lens is the ratio of its focal length to its “effective” diameter

$$F/\# = \frac{f}{D}$$

- Collect more light with small $F/\#$ lens but are more sensitive to aberrations
- Depth of field (DOF) = range of distances where the object (subject) is in “acceptable” focus (often in µm for microscopes)
- DOF increases with increasing #
- A lens with *F/#* = *f*/64 has larger DOF than *F/#* = *f*/1.4 [*F/#* → *f*/0.5 for microscopes]

OPTICS: THEORIES



Ray optics-Limit of wave optics when wavelength is infinitesimally small

Wave optics-provides a description of optical phenomena using scalar wave theory

Electromagnetic optics- provides most complete treatment of light within classical optics

Quantum optics-provides a quantum mechanical description of the electromagnetic theory

WAVE OPTICS

- Light propagates in the form of waves
- Can be described by a scalar (wave) function
- Does not completely describe some phenomena such as reflection and refraction, and polarization effects

POSTULATES OF WAVE OPTICS

- The wave equation

$$\boxed{\nabla^2 u(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2} = 0}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

wavefunction (real)

- Equation is linear->principle of superposition
- Describes how a wave behaves in *space* and *time*
- Approximately applicable if $n(\mathbf{r})$ and $c(\mathbf{r})$

POSTULATES OF WAVE OPTICS

- Optical intensity [Watts/cm^2]

$$\boxed{I(\mathbf{r},t) = 2\langle u^2(\mathbf{r},t)\rangle}$$

measurable

- Optical power [Watts]
- Optical energy [joules]

$$P(t) = \int_{\mathbf{A}} I(\mathbf{r},t)d\mathbf{A}$$

area normal to direction of propagation

$$E = \int_t P(t)dt$$

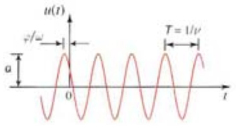
time interval

MONOCHROMATIC WAVES

generally, position dependent

$$u(\mathbf{r},t)=a(\mathbf{r})\cos[2\pi\upsilon t+\varphi(\mathbf{r})]$$

single frequency
(3x10^{11}-10^{16} Hz)



COMPLEX REPRESENTATION

- More convenient for studying optics

$$u(\mathbf{r},t)=\mathrm{Re}\{U(\mathbf{r},t)\}=\frac{1}{2}[U(\mathbf{r},t)+U^*(\mathbf{r},t)]$$

$$U(\mathbf{r},t)=\underbrace{a(\mathbf{r})\exp[j\varphi(\mathbf{r})]}_{\text{Complex amplitude, }U(\mathbf{r})}\exp(j2\pi\upsilon t)$$

Complex wavefunction

- $U(\mathbf{r},t)$ must also satisfy wave equation

OPTICAL INTENSITY (MONOCHROMATIC WAVE)

$$I(\mathbf{r})=|U(\mathbf{r})|^2$$

- averaged over a time longer than an optical period
- *I* is time independent for monochromatic waves

HELMHOLTZ EQUATION

$$\boxed{\nabla^2 U+k^2 U=0}$$

wavenumber, $k=\frac{2\pi\upsilon}{c}=\frac{\omega}{c}$

- Describes the wave's behavior in *space* only
- Convenient for studying many concepts

HELMHOLTZ EQUATION

$$\nabla^2 U + k^2 U = 0$$

- What are some solutions?
- Simplest solutions are plane wave, spherical wave

THE PLANE WAVE

$$U(\mathbf{r}) = A\exp(-j\mathbf{k}\cdot\mathbf{r}) = A\exp[-j(k_x x + k_y y + k_z z)]$$

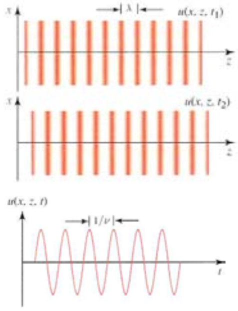
complex envelope

wavevector, $\mathbf{k} = (k_x, k_y, k_z)$

- Plugging $U(\mathbf{r})$ into Helmholtz equation gives
$$|\mathbf{k}| = k^2 = k_x^2 + k_y^2 + k_z^2$$
- Phase of $U(\mathbf{r})$ is given by $\arg\{U(\mathbf{r})\} = \arg\{A\} - \mathbf{k}\cdot\mathbf{r}$
- The wavefronts (surfaces of constant phase) obey
$$\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z = \enclose{circle}{2\pi q} + \arg\{A\}$$

THE PLANE WAVE

$$\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z = \enclose{circle}{2\pi q} + \arg\{A\}$$

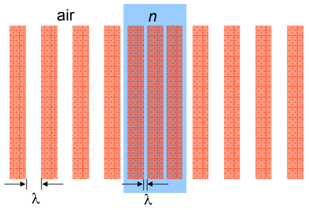


- Eq. describes parallel planes that are perpendicular to **k**
- Wavelength, $\lambda = 2\pi/k$ =c/ν [nm]
- Intensity (I=|A|^2) is constant everywhere---not realistic!

Image Source: Fundamentals of Photonics

WHAT DOES IT ALL MEAN?

- Given ν, and remembering that c_o is constant



$$c = \frac{c_o}{n},\quad \lambda = \frac{\lambda_o}{n},\quad k = nk_o$$

Image Source: Fundamentals of Photonics

THE SPHERICAL WAVE

- Solution to Helmholtz equation in spherical coordinates

$$U(\mathbf{r}) = \frac{A_o}{r}\exp(-jkr)$$

radial distance from origin

$$I(\mathbf{r}) = \frac{|A_o|^2}{r^2}$$

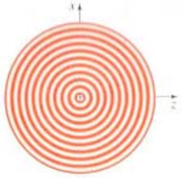


Image Source: Fundamentals of Photonics

PARAXIAL HELMHOLTZ EQUATION

$$\nabla_T^2 A - j2k\frac{\partial A}{\partial z} = 0$$

transverse Laplacian, $\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- Paraxial approximation of the Helmholtz equation
- Applies to paraxial waves, i.e., wavefront normals make small angles with z-axis
- Some solutions are the paraboloidal wave and the Gaussian beam

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THE PARABOLOIDAL WAVE

- Consider spherical wave at points close to z-axis and far from origin

$$U(\mathbf{r}) = \frac{A_o}{r}\exp(-jkr)$$

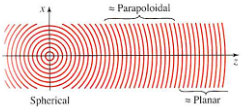
$r \approx z$

$$r \approx z + \frac{(x^2 + y^2)}{2z}$$

Fresnel approximation of a spherical wave

$$U(\mathbf{r}) \approx \frac{A_o}{z}\exp(-jkz)\exp\left[-jk\frac{x^2 + y^2}{2z}\right]$$

THE PARABOLOIDAL WAVE



- Far from origin spherical wave ~ paraboloidal wave
- Very far ~ plane wave
- Approximation is valid when

$$\frac{N_F \theta_m^2}{4} << 1$$

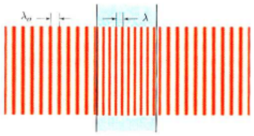
where, $N_F = \frac{a^2}{\lambda z}$

Radius of circle within which approx is valid

Fresnel number

Image Source: Fundamentals of Photonics

TRANSMISSION THROUGH A TRANSPARENT PLATE



- Complex amplitude transmittance

$$t(x,y)=\frac{U(x,y,d)}{U(x,y,0)}$$

- Inside plate, wave continues as a plane wave

- Thus, transmittance at output is

$$t(x,y)=\exp(-jnk_o d)$$

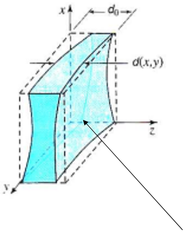
- Plate only introduces a phase shift

- For oblique incident angle,

$$t(x,y)=\exp(-jnk_o d\cos\theta_1)$$

Image Source: Fundamentals of Photonics

THIN TRANSPARENT PLATE (VARYING THICKNESS)



Thin transparent plate

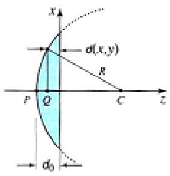
- For arbitrary paraxial wave input

$$t(x,y)\approx\exp(-jk_o d)\exp[-j(n-1)k_o d(x,y)]$$

- This is true if $(d_o/\lambda_o)\theta^2/2n<<1$ (paraxial approximation)

Image Source: Fundamentals of Photonics

TRANSMISSION THROUGH A THIN LENS



- At the point (*x*, *y*)

$$d(x,y)=d_o-\left[R-\sqrt{R^2-(x^2+y^2)}\right]$$

$$\approx d_o-\frac{x^2+y^2}{2R}$$

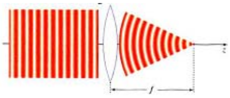
$$t(x,y)\approx\exp(-jk_o d)\exp\left[jk_o\frac{x^2+y^2}{2f}\right]$$

$$f=\frac{R}{n-1}$$

- Transforms planar wavefronts to paraboloidal wavefronts centered at *f*

Image Source: Fundamentals of Photonics

EXAMPLE



- *t*(*x*,*y*) for a thin lens is

$$t(x,y)\approx\exp(-jk_o d)\exp\left[jk_o\frac{x^2+y^2}{2f}\right]$$

- What does lens do to incident plane wave?

$$t(x,y)=\frac{U_2(x,y)}{U_1(x,y)}$$

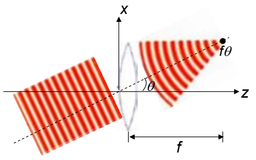
exp(-j*kz*)

$$\boxed{U_2(x,y)=\exp(-jkz)h_o\exp\left[jk_o\frac{x^2+y^2}{2f}\right]}$$

- Thus, lens transforms incoming wavefronts into paraboloidal wavefronts centered at (0,0,*f*)

Image Source: Fundamentals of Photonics

EXAMPLE



•For small input angle

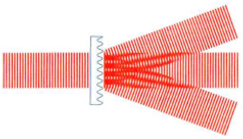
$$U_1(x,y) = \exp[-jk(z+\theta x)]$$

$$\boxed{U_2(x,y) = \exp(-jkz)h_o \exp\left[jk_o \frac{(x-f\theta)^2+y^2}{2f}\right]}$$

•Transforms input to paraboloidal wave centered at (*f* θ,0,*f*)

Image Source: Fundamentals of Photonics

DIFFRACTION GRATINGS



- Periodically modulates amplitude or phase of input wave
- Here incident plane wave is split into multiple plane waves traveling in different directions

In paraxial approximation,

$$\theta_q = \theta_i + q\frac{\lambda}{\Lambda}$$

Grating equation

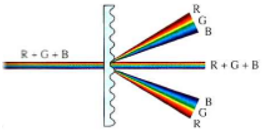
incident angle

diffraction order

grating period

Image Source: Fundamentals of Photonics

DIFFRACTION GRATINGS



- More general grating equation

$$\sin\theta_q = \sin\theta_i + q\frac{\lambda}{\Lambda}$$

- λ dependent
- Used as filters and spectrum analyzers

Image Source: Fundamentals of Photonics

GRADED-INDEX COMPONENTS

Optical Component	Transverse Variation in thickness of material
Prism	Linear
Lens	Quadratic
Diffraction grating	Periodic

• GRIN components can be designed to produce the same effects

$$t(x,y) = \exp(-jn(x,y)k_o d_o)$$

INTERFERENCE

INTERFERENCE

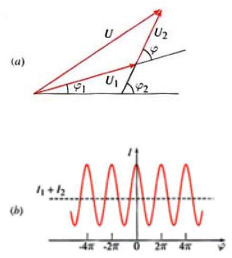
- Let’s consider the superposition of two monochromatic waves

$$U(\mathbf{r}) = U_1(\mathbf{r}) + U_2(\mathbf{r})$$

$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^*$$

$$U_1 = \sqrt{I_1}\exp(j\varphi_1) \quad , \qquad U_2 = \sqrt{I_2}\exp(j\varphi_2)$$

INTERFERENCE



Interference equation

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\varphi}$$

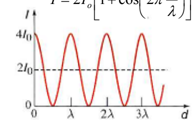
$$\varphi = \varphi_2 - \varphi_1$$

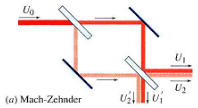
$$I = 2I_o(1+\cos\varphi) = 4I_o\cos^2\left(\frac{\varphi}{2}\right)$$

Image Source: Fundamentals of Photonics

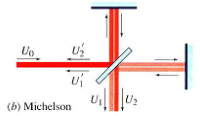
INTERFEROMETERS

$$I = 2I_o\left[1+\cos\left(2\pi\frac{d}{\lambda}\right)\right]$$

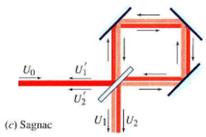




(a) Mach-Zehnder



(b) Michelson



(c) Sagnac

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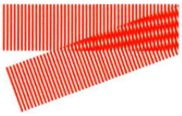
INTERFERENCE OF TWO OBLIQUE PLANE WAVES



$$I = 2I_o\left[1+\cos(k\sin\theta x)\right]$$

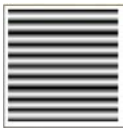


Image Source: **Fundamentals of Photonics**

BREAK

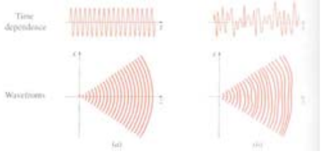
STATISTICAL OPTICS

DEFINITIONS

- Study of random light
- Occurs b/c of fluctuations of the light either due to the source or the medium that it propagates through
- Examples:

1) Light scattered from rough surfaces (imparts random variations in the wavefront)
2) Natural light radiated by a hot object is made up of A LOT of atoms, each radiating at different frequencies and phases

DEFINITIONS



- Deterministic light is called “coherent”
- Ex: monochromatic plane wave
- Random light, the behavior of the wavefunction *U*(**r**,t) with respect to position and time is not (completely) predictable
- Statistical methods are used to describe random light

Image Source: **Fundamentals of Photonics**

WHAT CAN WE DO?

- Which is more “intense”?
- Which has a “faster” fluctuating envelope?

Image Source: Fundamentals of Photonics

OPTICAL INTENSITY

- Recall that for deterministic (coherent) light

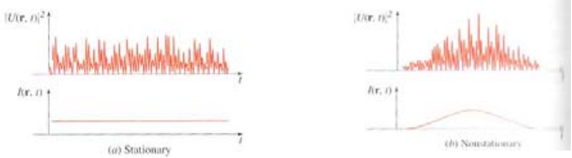
$$I(\mathbf{r},t)=|U(\mathbf{r},t)|^2$$

- For random light, *U*(**r**,t) fluctuates with position and time

Average intensity $I(\mathbf{r},t)=\left\langle \underbrace{|U(\mathbf{r},t)|^2}_{\text{Instantaneous intensity}} \right\rangle$

Ensemble average over many instances

STATIONARY VS. NONSTATIONARY



- (Statistically) Stationary - *I*(**r**,t) = *I*(**r**) is time independent
  Ex: light bulb driven by constant electric current

$$I(\mathbf{r})=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|U(\mathbf{r},t)|^2dt$$

- (Statistically) Nonstationary – *I*(**r**,t) is not time independent
  Ex: light bulb driven by pulse of electric current

Image Source: Fundamentals of Photonics

TEMPORAL COHERENCE

- Consider stationary light at fixed position ***r*** :
  U(***r***,*t*) → *U*(*t*) and *I*(***r***)→ *I*

- We characterize *U*(*t*) by a time scale that represents the memory of the random fluctuation

- Quantitatively this is represented by the autocorrelation function *G*(τ)

TEMPORAL COHERENCE FUNCTION

- Autocorrelation(temporal coherence function)

$$\boxed{G(\tau)=\langle U^*(t)U(t+\tau)\rangle}$$

$$G(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}U^*(t)U(t+\tau)dt$$

$$\boxed{I=G(0)}$$

- Contains info about intensity and coherence (degree of correlation)

DEGREE OF TEMPORAL COHERENCE

- Complex degree of temporal coherence

$$\boxed{g(\tau)=\frac{G(\tau)}{G(0)}=\frac{\langle U^*(t)U(t+\tau)\rangle}{\langle U^*(t)U(t)\rangle}} \qquad 0\le|g(\tau)|\le 1$$

Normalized autocorrelation

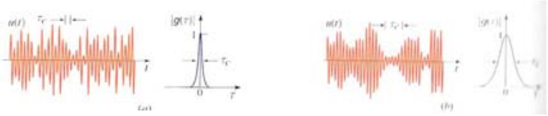
- For monochromatic wave, $U(t)=A\exp(j\omega_o t)$

$$\downarrow$$

$$g(\tau)=A\exp(j\omega_o\tau)\longrightarrow|g(\tau)|=1$$

- Tells you how much two points in time are correlated (for a fixed position)

COHERENCE TIME



- Within τ_c, wave is predictable; At times greater than τ_c, one cannot predict amplitude and phase of wave

$$\boxed{\tau_c=\int_{-\infty}^{\infty}|g(\tau)|^2d\tau}$$

Commonly used definition

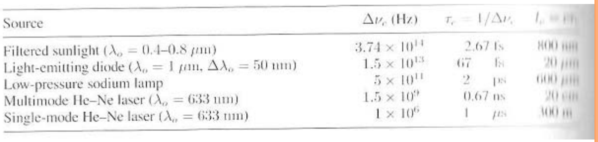
$\tau_c=\infty$ Monochromatic wave

Image Source: Fundamentals of Photonics

COHERENCE LENGTH

- Light can be considered coherent if distance l_c is >> than all optical path-length differences encountered in an optical system

$$\boxed{l_c=c\tau_c}$$



Source	$\Delta\nu_c$ (Hz)	$\tau_c=1/\Delta\nu_c$	$l_c=c\tau_c$
Filtered sunlight ($\lambda_o=0.4$–0.8 μm)	3.74×10^{14}	2.67 fs	800 nm
Light-emitting diode ($\lambda_o=1$ μm, $\Delta\lambda_o=50$ nm)	1.5×10^{13}	67 fs	20 μm
Low-pressure sodium lamp	5×10^{11}	2 ps	600 μm
Multimode He–Ne laser ($\lambda_o=633$ nm)	1.5×10^{9}	0.67 ns	20 cm
Single-mode He–Ne laser ($\lambda_o=633$ nm)	1×10^{6}	1 μs	300 m

Image Source: Fundamentals of Photonics

COMPLEX DEGREE OF COHERENCE

- Spatial and temporal fluctuations of *U*(***r***,*t*) can be described by cross-correlation between *U*($\boldsymbol{r}_1$,*t*) and U($\boldsymbol{r}_2$,*t*)

$$G(\mathbf{r}_1,\mathbf{r}_2,\tau)=\left\langle U^*(r_1,t)U(r_2,t+\tau)\right\rangle$$

Mutual coherence function

$$\downarrow$$

$$g(\mathbf{r}_1,\mathbf{r}_{2,}\tau)=\frac{G(\mathbf{r}_1,\mathbf{r}_{2,}\tau)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}} \qquad 0\le\left|g(\mathbf{r}_1,\mathbf{r}_2,\tau)\right|\le 1$$

- Measure of coherence between 2 positions at 2 different times

---

MUTUAL INTENSITY

- To examine the spatial correlation of light

$$G(\mathbf{r}_1,\mathbf{r}_2,0)=G(\mathbf{r}_1,\mathbf{r}_2)=\left\langle U^*(r_1,t)U(r_2,t)\right\rangle$$

$$g(\mathbf{r}_1,\mathbf{r}_{2,}0)=g(\mathbf{r}_1,\mathbf{r}_2)=\frac{G(\mathbf{r}_1,\mathbf{r}_2)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}$$

Normalized mutual intensity

- Measure of spatial coherence between 2 positions for 0 time delay

---

COHERENCE AREA

- Represents the spatial extent of $\left|g(\mathbf{r}_1,\mathbf{r}_2)\right|$ as a function of $\mathbf{r}_1$ for a fixed $\mathbf{r}_2$
- If size of aperture in optical system is < than coherence area ($\left|g(\mathbf{r}_1,\mathbf{r}_2)\right|\approx 1$) then light may be considered coherent; if vice-versa, light is considered incoherent

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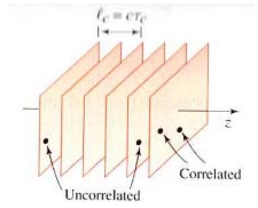
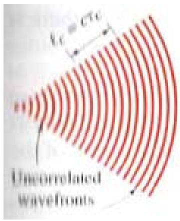
PARTIAL COHERENCE





Plane wave

Spherical wave

Image Source: Fundamentals of Photonics

INTERFERENCE OF PARTIALLY COHERENT LIGHT

partially coherent waves 1 and 2

$$\boxed{I=\left\langle \left|U_1+U_2\right|^2\right\rangle = I_1+I_2+2\sqrt{I_1I_2}\left|g_{12}\right|\cos\varphi}$$

- If both waves are completely correlated w/ $g_{12}=\exp(j\varphi)$ then:

$$\left|g_{12}\right|=1$$

and *I* is given by usual expression we know

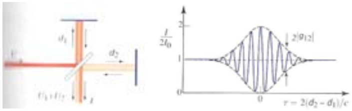
$$g_{12}=\frac{\left\langle U_1^*U_2\right\rangle}{\sqrt{I_1I_2}}$$

Phase of g_{12}

$$\boxed{V=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=\frac{2\sqrt{I_1I_2}}{I_1+I_2}\left|g_{12}\right|}$$

- If both waves are completely uncorrelated w/ $g_{12}=0$ then:

$$I=I_1+I_2$$



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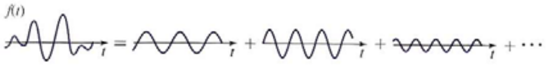
BREAK

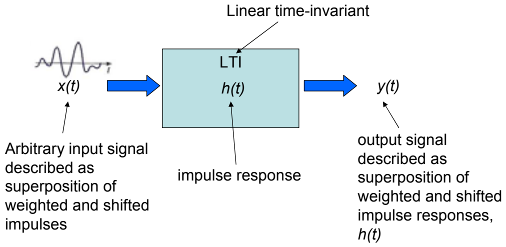
FOURIER OPTICS

BACKGROUND: FOURIER ANALYSIS



- Arbitrary function can be represented as a weighted superposition of harmonic functions
- Frequencies are orthogonal

FOURIER ANALYSIS (TIME-DOMAIN APPROACH)



Linear time-invariant

LTI

h(t)

x(t)

y(t)

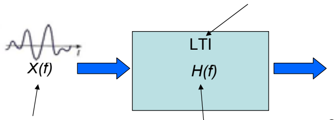
Arbitrary input signal described as superposition of weighted and shifted impulses

impulse response

output signal described as superposition of weighted and shifted impulse responses, *h(t)*

$$y(t)=x(t)*h(t)$$

FOURIER ANALYSIS (FREQ-DOMAIN APPROACH)



Linear time-invariant

LTI

H(f)

X(f)

Y(f)

Arbitrary input signal described as superposition of weighted sinusoids (harmonic signals)

Frequency response or Transfer function

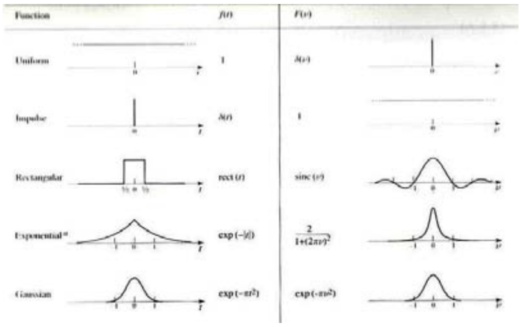
output signal described as superposition of weighted and shifted response *H(f)*

$$Y(f) = X(f) \cdot H(f)$$

FOURIER ANALYSIS

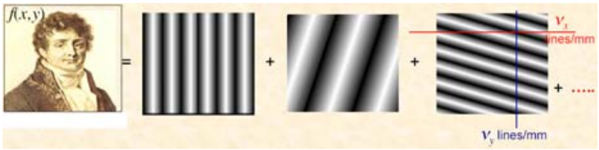
- *Y(f)* is the Fourier transform of *y(t)*
- *y(t) is the Inverse Fourier transform of Y(f)*

SOME COMMON TRANSFORM PAIRS



Function	$f(t)$	$F(\nu)$
Uniform	1	$\delta(\nu)$
Impulse	$\delta(t)$	1
Rectangular	rect (t)	sinc (ν)
Exponential^a	$\exp(-\lvert t\rvert)$	$\frac{2}{1+(2\pi\nu)^2}$
Gaussian	$\exp(-\pi t^2)$	$\exp(-\pi\nu^2)$

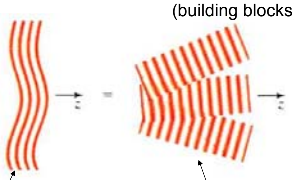
INTRODUCTION



- Image can be decomposed into weighted sum of harmonic functions
- Orthogonal spatial frequencies
- Different complex amplitudes

INTRODUCTION

Arbitrary optical wave = Superposition of plane waves
(building blocks)



Composed of various spatial frequencies

Each plane wave has unique spatial frequency ν

SPATIAL FREQUENCY

Spatial angular frequency [radians/mm]

$$\nu_x = \frac{k_x}{2\pi} \qquad \nu_y = \frac{k_y}{2\pi}$$

[cycles/mm]

- Tells us how often structure (e.g., in an image) repeats per unit distance

---

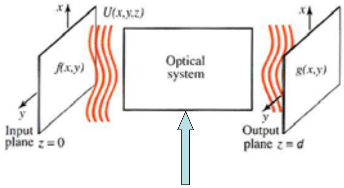
THE EFFECT OF SPATIAL FREQUENCIES

HPF

LPF

[courtesy: http://www.icaen.uiowa.edu/~dip/LECTURE/LinTransforms.html]

LINEAR-SYSTEMS APPROACH



Linear system can be described in either of two pictures

Impulse response function
-response to impulse or point at input

Transfer function
-response to spatial harmonic function

SPATIAL HARMONIC FUNCTION

- Consider plane wave at arbitrary plane

$$U(x,y,z)=A\exp\left[-j\left(k_x x+k_y y+k_z z\right)\right]$$

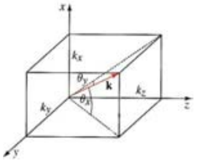
- At z=0 plane

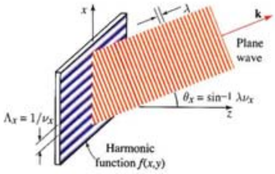
$$U(x,y,0)=A\exp\left[-j\left(k_x x+k_y y\right)\right]=f(x,y)$$

$$=A\exp\left[-j2\pi\left(\frac{k_x}{2\pi}x+\frac{k_y}{2\pi}y\right)\right]$$

$$=A\exp\left[-j2\pi\left(\nu_x x+\nu_y y\right)\right]$$

ANGLES





$$U(x,y,0)=A\exp\left[-j2\pi\left(\nu_x x+\nu_y y\right)\right]=f(x,y)$$

↓

consistent w/ plane wave traveling at angles

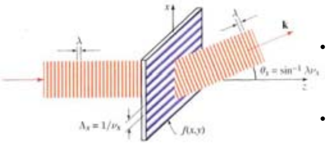
$$\theta_x=\sin^{-1}\lambda\nu_x \qquad \theta_y=\sin^{-1}\lambda\nu_y$$

in paraxial approximation

$$\theta_x\approx\lambda\nu_x\,,\qquad \theta_y\approx\lambda\nu_y$$

notice wavelength dependence

SPATIAL SPECTRAL ANALYSIS



- Plane wave is deflected up $f(x,y)=\exp[-j2\pi(\nu_x x+\nu_y y)]$
- Plane wave is deflected down for $f(x,y)=\exp[+j2\pi(\nu_x x+\nu_y y)]$
- Interference phenomenon where two points separated by Λ interfere constructively for pathlength difference λ

Image Source: Fundamentals of Photonics

EXAMPLES

f(x,y)	Response
$\cos(2\pi\nu_x x)$ $=\frac{1}{2}[\exp(-j2\pi\nu_x x)+\exp(+j2\pi\nu_x x)]$	• Incident plane wave is transformed into components bent both upward and downward $\pm\sin^{-1}\lambda\nu_x$
$1+\cos(2\pi\nu_y y)$	• Incident plane wave is transformed into components bent left and right as well as a portion that goes through

MATHEMATICAL DESCRIPTION

$$f(x,y)=\iint_{-\infty} F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$$

superposition of harmonic functions

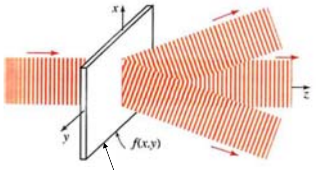
$$U(x,y,z)=\iint_{-\infty} \overbrace{F(\nu_x,\nu_y)\exp[-j(2\pi\nu_x x+2\pi\nu_y y)]}^{f(x,y)}\exp(-jk_z z)d\nu_x d\nu_y$$

superposition of plane waves

$$k_z=\pm\sqrt{k^2-k_x^2-k_z^2}=2\pi\sqrt{\lambda^{-2}-\nu_x^2-\nu_y^2}$$

$$\boxed{f(x,y)\overset{FT}{\leftrightarrow}F(\nu_x,\nu_y)}$$

EFFECT OF THIN OPTICAL ELEMENT

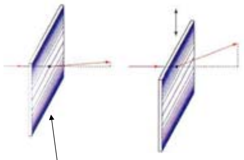


Thin optical element

- Transmitted light can be written as superposition of plane ways provided that $\boxed{\nu_x^2+\nu_y^2\leq\lambda^{-2}}$

Image Source: Fundamentals of Photonics

EXAMPLE

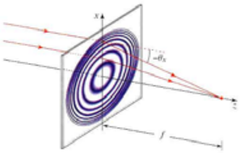


$$f(x,y)=\exp\left(\frac{j\pi x^2}{\lambda f}\right)$$

- Phase shift 2πϕ(x,y) is introduced
- $\phi(x,y)=-x^2/2\lambda f$
- Wave is deflected at position (*x*,*y*) by
  $$\theta_x=\sin^{-1}\left(\underbrace{\frac{\lambda\partial\phi}{\partial x}}_{\nu_x}\right)=\sin^{-1}\left(\frac{-x}{f}\right)$$
- In paraxial limit
  $$\theta_x\approx-\frac{x}{f}$$
  ← different x-values yield different angles

Image Source: Fundamentals of Photonics

EXAMPLE

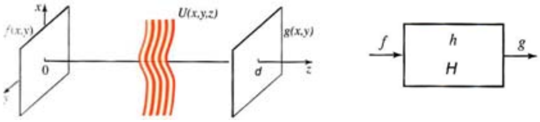


$$f(x,y)=\exp\left(\frac{j\pi x^2}{\lambda f}\right)$$

- Illuminate transparency from previous example w/ plane wave
- Each part of wave is deflected by a different angle
- Transparency acts as a cylindrical lens
- Transparency acts as a spherical lens for $f(x,y)=\exp\left(\frac{j\pi(x^2+y^2)}{\lambda f}\right)$

Image Source: Fundamentals of Photonics

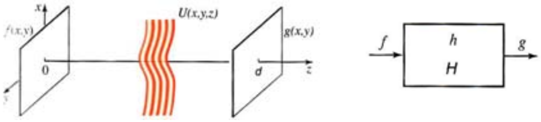
TRANSFER FUNCTION OF FREE SPACE



- We use systems analysis to determine the output *g*(x,*y*)=U(x,y,d), given input *f*(*x*,*y*)=U(x,y,0)
- System is linear shift-invariant (LSI) because of Helhmoltz equation and invariance of free space to displacement

Image Source: Fundamentals of Photonics

TRANSFER FUNCTION OF FREE SPACE

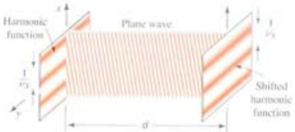


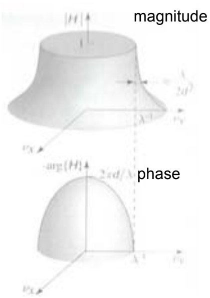
$$g(x,y)=f(x,y)\cdot H(\nu_x,\nu_y)$$

$$\boxed{H(\nu_x,\nu_y)=\exp\left[-j2\pi d\sqrt{\lambda^{-2}-\nu_x^2-\nu_y^2}\right]}$$

Image Source: Fundamentals of Photonics

TRANSFER FUNCTION OF FREE SPACE





- Harmonic function with spatial frequencies $\nu_x^2+\nu_y^2\leq\lambda^{-2}$ will have its magnitude go unaltered, but it will undergo a spatial phase shift
- For $\nu_x^2+\nu_y^2>\lambda^{-2}$, term under square root leads to a real exponential $\exp\left[-2\pi d\sqrt{\nu_x^2+\nu_y^2-\lambda^{-2}}\right]$ ← attenuation factor→ evanescent wave

Image Source: Fundamentals of Photonics

TRANSFER FUNCTION OF FREE SPACE

- For *d*>>λ, attenuation factor drops sharply for spatial frequencies that slightly exceed λ^{-1}
- λ^{-1} [cycles/mm] is ~ the spatial bandwidth of free space
- For any image, features with details finer than λ (spatial frequencies greater than λ^{-1}) cannot be transmitted by light of wavelength λ for *d*>>λ
- Conceptually, this is the difference between near field and far field microscopy

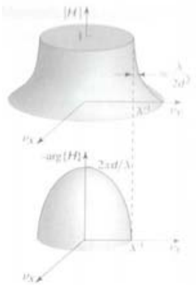


Image Source: Fundamentals of Photonics

FRESNEL APPROXIMATION

- For $\nu_x^2+\nu_y^2 << \lambda^{-2}$, we use paraxial approximation such that $\theta_x \approx \lambda \nu_x$ and $\theta_y \approx \lambda \nu_y$
- Let $\theta^2 = \theta_x^2+\theta_y^2 \approx \lambda^2(\nu_x^2+\nu_y^2)$, then phase factor becomes

$$2\pi d\sqrt{\lambda^{-2}-\nu_x^2-\nu_y^2} = 2\pi\frac{d}{\lambda}\sqrt{1-\theta^2}$$

Taylor series expansion

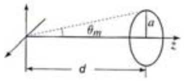
$$= 2\pi\frac{d}{\lambda}\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{2}-\cdots\right)$$

FRESNEL APPROXIMATION

$$\boxed{H(\nu_x,\nu_y) \approx H_o \exp\left[j\pi\lambda d\left(\nu_x^2+\nu_y^2\right)\right]}$$

exp(-*jkd*)

- Phase is a quadratic function of the spatial frequencies
- Approximation holds if $\boxed{N_F\frac{\theta_m^2}{4} << 1}$



$$N_F = \frac{a^2}{\lambda d}$$

- For, *a*=1 cm, *d*=100 cm, λ=500 nm, $N_F\theta^2/4$=.005<<1

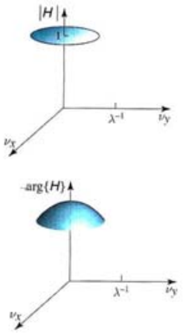


Image Source: Fundamentals of Photonics

INPUT-OUTPUT RELATION

- Given input *f*(*x*,*y*) we can determine output using Fourier transform (spatial-frequency) approach

1. Determine complex envelope of input plane-wave components

$$F(\nu_x,\nu_y) = \iint_{-\infty} f(x,y)\exp\left[j2\pi(\nu_x x+\nu_y y)\right]dxdy \overset{FT}{\leftrightarrow} f(x,y)$$

2. Determine complex envelope of output plane-wave components by $H(\nu_x,\nu_y)F(\nu_x,\nu_y)$
3. Determine complex amplitude *g*(*x*,*y*) by summing individual contributions

inverse FT

$$\boxed{g(x,y) = \iint_{-\infty} H(\nu_x,\nu_y)F(\nu_x,\nu_y)\exp\left[-j2\pi(\nu_x x+\nu_y y)\right]d\nu_x d\nu_y}$$

INPUT-OUTPUT RELATION

- Using Fresnel approximation for *H*(ν_x,ν_y)

$$g(x,y) = H_o\iint_{-\infty} F(\nu_x,\nu_y)\exp\left[j\pi\lambda d\left(\nu_x^2+\nu_y^2\right)\right]\exp\left[-j2\pi(\nu_x x+\nu_y y)\right]d\nu_x d\nu_y$$

IMPULSE RESPONSE FUNCTION (FREE SPACE)

- In the Fresnel approximation we define

$$\boxed{h(x,y) \approx h_o \exp\left[-jk\frac{x^2+y^2}{2d}\right]}$$

($j/\lambda d$)exp($-jkd$)

Impulse response function → response $g(x,y)$ when input is a point at the origin
(point-spread function)

- Each input point generates a paraboloidal wave that is summed at output

FREE-SPACE PROPAGATION (CONVOLUTION)

- Input-output can be related using space-domain approach
- Think of *f*(*x*,*y*) as superposition of different points (delta functions)

$$g(x,y) = \iint_{-\infty} f(x',y')h(x-x',y-y')dx'dy'$$

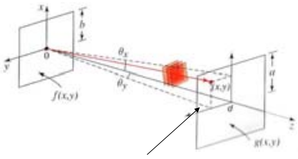
- In Fresnel approximation,

$$\boxed{g(x,y) = h_o\iint_{-\infty} f(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy'}$$

OPTICAL FOURIER TRANSFORM

- Light can be used to determine FT of 2-D function *f*(*x*,*y*)
- Design a transparency with amplitude transmittance *f*(*x*,*y*) and then illuminate with a plane wave
- FT of *f*(*x*,*y*) is then determined either after light propagates a really long distance *d* or by using a lens

FOURIER TRANSFORM (FAR FIELD)



•Can spatially resolve each plane wave contribution if *d* is long enough

$$\boxed{g(x,y) \approx h_o F\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)}$$

- This is valid under Fraunhofer approximation which requires that

$\boxed{N_F << 1}$ and $\boxed{N_F{}' << 1}$

$$N_F = \frac{a^2}{\lambda d} \text{ and } N_F{}' = \frac{b^2}{\lambda d}$$

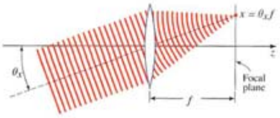
describes largest circular region in observation plane

describes largest circular region where points lie in object plane

- More difficult to satisfy than Fresnel approximation

Image Source: Fundamentals of Photonics

FOURIER TRANSFORM (USING LENS)

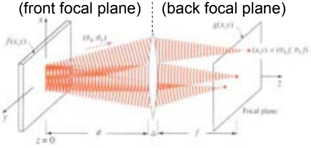


- Lens maps each direction (θ_x, θ_y) to a single point $(\theta_x f, \theta_y f)$
- Allows us to separate different plane wave contributions

Image Source: Fundamentals of Photonics

FOURIER TRANSFORM (USING LENS)

(front focal plane) (back focal plane)



- Complex amplitude *g*(x,y) at point (*x*,*y*) in output plane is (using Fresnel approximation)

$$g(x,y) = h_l \exp\left[j\pi \frac{(x^2+y^2)(d-f)}{\lambda f^2} \right] F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

$H_o h_o$

Fourier transform of (*x*,*y*) evaluated at spatial frequencies ν_x and ν_y

Image Source: Fundamentals of Photonics

FOURIER TRANSFORM (USING LENS)

- Intensity of light at back focal plane (independent of input distance *d)*

$$I(x,y) = \frac{1}{(\lambda f)^2} \left| F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \right|^2$$

- For *d*=*f*, we get 2-*f* system

$$g(x,y) = h_l F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

(*j* /λ*f*)exp(-*j*2*kf*)

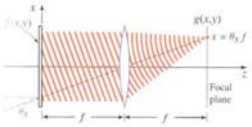


Image Source: Fundamentals of Photonics

BREAK

DIFFRACTION

DIFFRACTION

Diffraction patterns of the teeth of a saw -would be perfect geometric shadow if λ→ 0 (as in ray optics)

- Relates to the obstruction of a wave by an obstacle
- Obstruction could be bending of an optical wave by edges or when passing through an aperture (of appropriate size)
- Amplitude or phase of wave is altered
- Interference phenomenon

Image Source: Fundamentals of Photonics

DIFFRACTION OF A WAVE BY A SLIT

- Occurs for any type of wave phenomenon
- Effect is more pronounced as the size of the slit approaches the λ of the wave

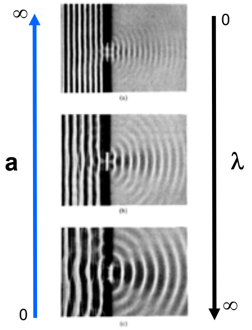
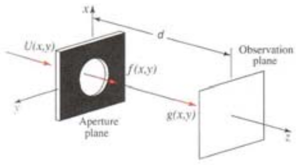


Image Source: Optics

BASIC APPROACH



- Assumption: Incident wave passes through unaltered within aperture; reduces to 0 on opaque part of aperture

$$f(x,y)=U(x,y)p(x,y)$$

$$p(x,y)=\begin{cases}1, & \text{inside the aperture}\\ 0, & \text{outside the aperture}\end{cases}$$

aperture (pupil) function

- Diffraction pattern is intensity at the observation plane

$$I(x,y)=|g(x,y)|^2$$

Fraunhofer (far-field) diffraction
or
Fresnel (near-field) diffraction

Image Source: Fundamentals of Photonics

FRAUNHOFER DIFFRACTION

- Used when light (free-space) propagation (beyond aperture) is described by Fraunhofer approximation
- Valid when $N_F{}' = \frac{b^2}{\lambda d} << 1$

Largest radial distance within aperture

- For $U(x,y)=\sqrt{I_i}$, $f(x,y)=\sqrt{I_i}\,p(x,y)$

$$\boxed{g(x,y)\approx\sqrt{I_i}\,h_o P\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)}$$

FT of *p*(*x*,*y*)

FRAUNHOFER DIFFRACTION

$$P\left(\nu_x,\nu_y\right)=\iint\limits_{-\infty} p(x,y)\exp\left[j2\pi\left(\nu_x x+\nu_y y\right)\right]dxdy \overset{\text{FT}}{\leftrightarrow} p(x,y)$$

$$\boxed{I(x,y)=\frac{I_i}{(\lambda d)^2}\left|P\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)\right|^2}$$

Diffraction pattern

$$\boxed{I(x,y)=\frac{I_i}{(\lambda f)^2}\left|P\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)\right|^2}$$

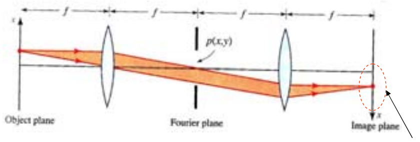
using a lens

FRESNEL DIFFRACTION

- Used when light (free-space) propagation (beyond aperture) is described by Fresnel approximation

$$\boxed{I(x,y)=\frac{I_i}{(\lambda d)^2}\left|\iint\limits_{-\infty} p(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy'\right|^2}$$

IMAGE FORMATION

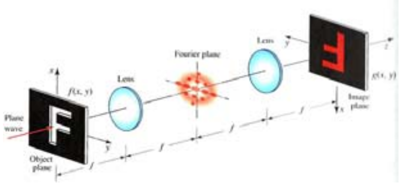


inverted coordinate system

- 4-*f* system (a.k.a. imaging system)
- Fourier transform taken twice
- Second lens performs “inverse” Fourier transform since coordinate system is inverted
- Perfect replica of object obtained

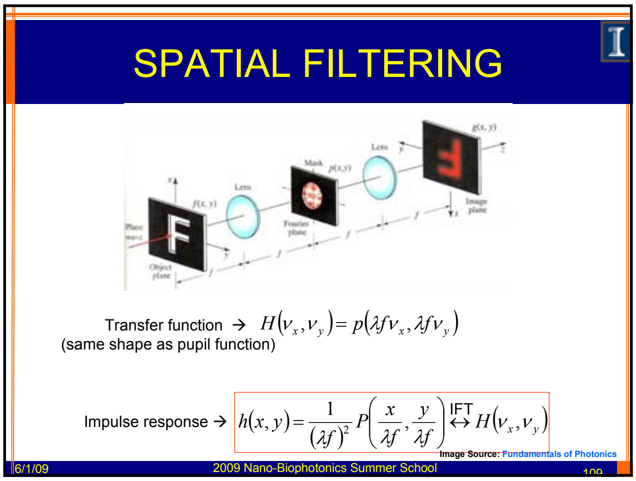
Image Source: Fundamentals of Photonics

4-*f* SYSTEM



- In Fourier plane, individual spatial frequency components are separated
- Allows for spatial filtering (physical process)

Image Source: Fundamentals of Photonics

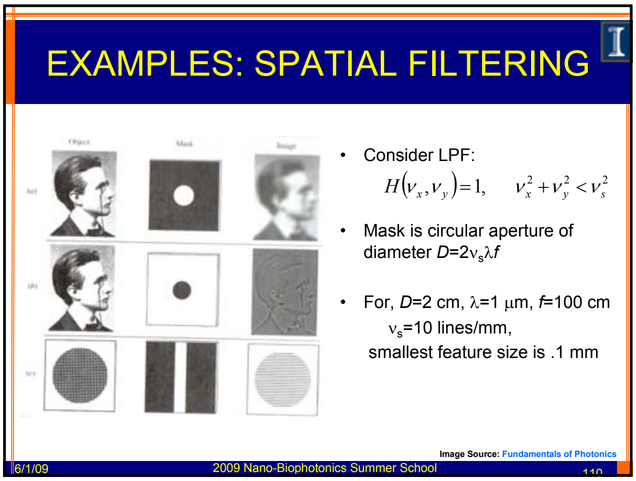


SPATIAL FILTERING

Transfer function → $H(\nu_x,\nu_y)=p(\lambda f\nu_x,\lambda f\nu_y)$
(same shape as pupil function)

Impulse response → $\boxed{h(x,y)=\frac{1}{(\lambda f)^2}P\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)\overset{\text{IFT}}{\leftrightarrow}H(\nu_x,\nu_y)}$

Image Source: Fundamentals of Photonics



EXAMPLES: SPATIAL FILTERING

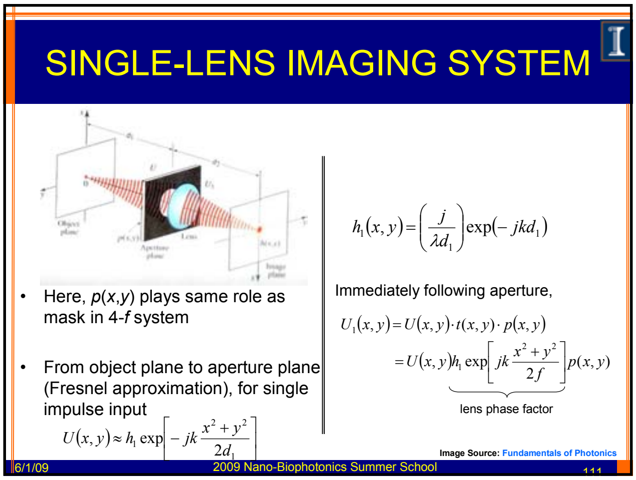
- Consider LPF:

  $H(\nu_x,\nu_y)=1,\qquad \nu_x^2+\nu_y^2<\nu_s^2$

- Mask is circular aperture of diameter *D*=2$\nu_s\lambda f$

- For, *D*=2 cm, λ=1 μm, *f*=100 cm
  ν_s=10 lines/mm,
  smallest feature size is .1 mm

Image Source: Fundamentals of Photonics



SINGLE-LENS IMAGING SYSTEM

- Here, *p*(*x*,*y*) plays same role as mask in 4-*f* system

- From object plane to aperture plane (Fresnel approximation), for single impulse input

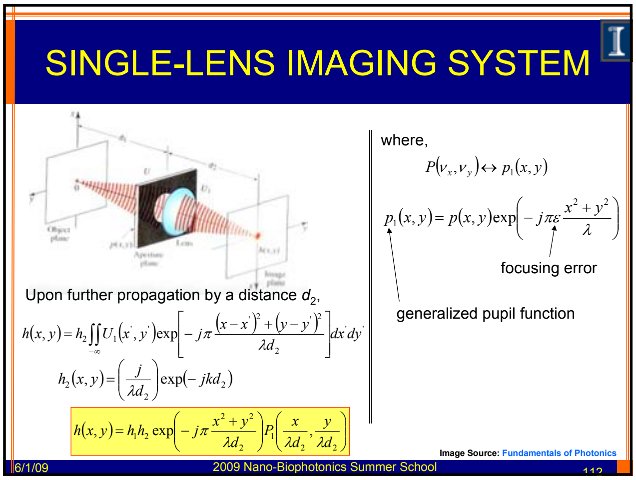
  $U(x,y)\approx h_1\exp\left[-jk\frac{x^2+y^2}{2d_1}\right]$

$$h_1(x,y)=\left(\frac{j}{\lambda d_1}\right)\exp(-jkd_1)$$

Immediately following aperture,

$$U_1(x,y)=U(x,y)\cdot t(x,y)\cdot p(x,y)$$
$$=U(x,y)\underbrace{h_1\exp\left[jk\frac{x^2+y^2}{2f}\right]}_{\text{lens phase factor}}p(x,y)$$

Image Source: Fundamentals of Photonics



SINGLE-LENS IMAGING SYSTEM

Upon further propagation by a distance *d*$_2$,

$$h(x,y)=h_2\iint_{-\infty}U_1(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d_2}\right]dx'dy'$$

$$h_2(x,y)=\left(\frac{j}{\lambda d_2}\right)\exp(-jkd_2)$$

$$\boxed{h(x,y)=h_1h_2\exp\left(-j\pi\frac{x^2+y^2}{\lambda d_2}\right)P_1\left(\frac{x}{\lambda d_2},\frac{y}{\lambda d_2}\right)}$$

where,

$$P(\nu_x,\nu_y)\leftrightarrow p_1(x,y)$$

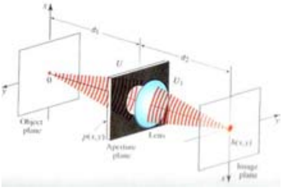
$$p_1(x,y)=p(x,y)\exp\left(-j\pi\varepsilon\frac{x^2+y^2}{\lambda}\right)$$

focusing error

generalized pupil function

Image Source: Fundamentals of Photonics

SINGLE-LENS IMAGING SYSTEM



$$h(x,y) = h_1 h_2 \exp\left(-j\pi \frac{x^2+y^2}{\lambda d_2}\right) P_1\left(\frac{x}{\lambda d_2}, \frac{y}{\lambda d_2}\right)$$

assume <<1

$$h(x,y) = h_1 h_2 P_1\left(\frac{x}{\lambda d_2}, \frac{y}{\lambda d_2}\right)$$

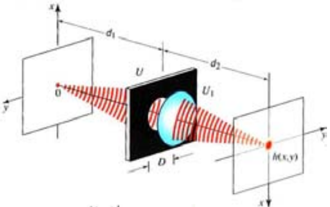
Now, for perfect focusing, *$p_1(x,y)=p(x,y)$*

$$\boxed{h(x,y) \approx h_1 h_2 P\left(\frac{x}{\lambda d_2}, \frac{y}{\lambda d_2}\right)}$$

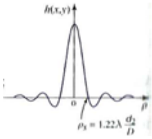
similar result as with 4-*f* system

Image Source: Fundamentals of Photonics

SINGLE-LENS IMAGING SYSTEM



$$U(x,y) \approx h_1 \exp\left[-jk\frac{x^2+y^2}{2d_1}\right]$$

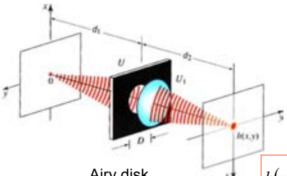


$$U_1(x,y) = U(x,y) h_1 \exp\left[jk\frac{x^2+y^2}{2f}\right] p(x,y)$$

$$h_1(x,y) = \left(\frac{j}{\lambda d_1}\right)\exp(-jkd_1)$$

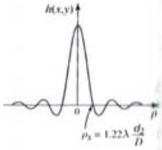
Image Source: Fundamentals of Photonics

EXAMPLE



$$p(x,y) = \begin{cases} 1, & \rho = \sqrt{x^2+y^2} \le D/2 \\ 0, & \text{outside the aperture} \end{cases}$$

Airy disk



$$\boxed{h(x,y) = h(0,0)\frac{2J_1(\pi D\rho/\lambda d_2)}{\pi D\rho/\lambda d_2}, \quad \rho = \sqrt{x^2+y^2}}$$

$$|h(0,0)| = \left(\pi D^2 / 4\lambda^2 d_1 d_2\right)$$

For, $d_1 = \infty, \quad d_2 = f$

$$\boxed{\rho_s = 1.22\lambda F_\#}$$

Smaller $F_\#$ (larger aperture) has better image quality

Measure of size of blur of circle

Image Source: Fundamentals of Photonics

BREAK

MICROSCOPY