ECE-656: Fall 2009

Lecture 18: Strong magnetic fields

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Hall and integer quantum Hall effect

Hall effect

$V_H \sim B$
V_x independent of B

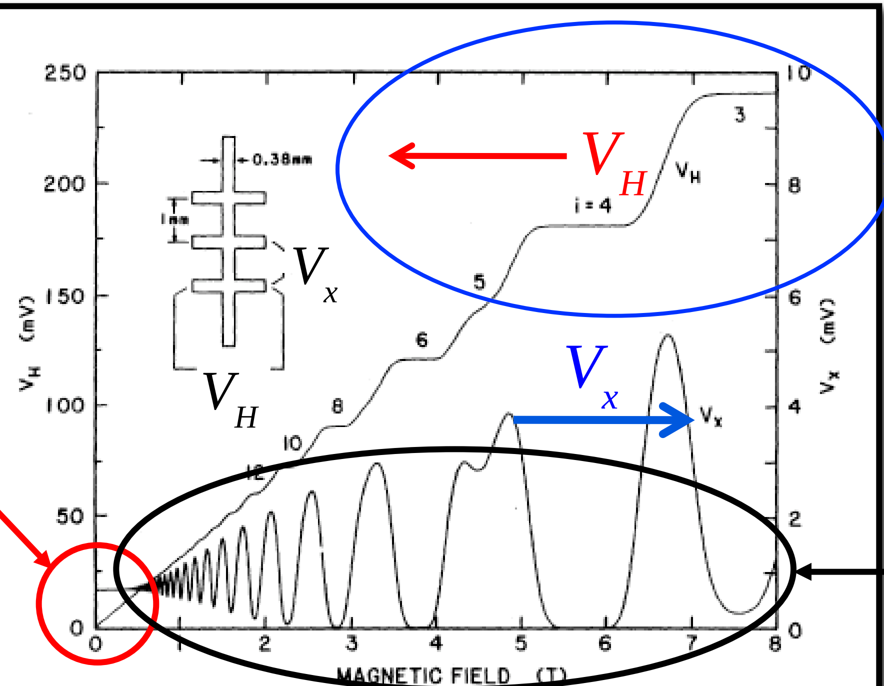


Fig. 1. Recording of V_H and V_x versus magnetic field for a GaAs device cooled to 1.2 K. The current is 25.5 μA.

Quantum Hall effect

Longitudinal magneto-resistance

M.E. Cage, R.F. Dziuba, and B.F. Field, “A Test of the Quantum Hall Effect as a Resistance Standard,” *IEEE Trans. Instrumentation and Measurement*,” Vol. IM-34, pp. 301-303, 1985

outline

1) **Magnetoconductivity tensor**

2) Resistivity tensor

3) Strong B-fields: Landau levels

4) Shubnikov-DeHaas Oscillations and QHE

5) Summary

B-field dependent DD equation

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = \left(-q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}\right)\tau = m^* \vec{\upsilon}$$

$$\vec{\upsilon} = -\frac{q\tau}{m^*}\vec{\mathcal{E}} - \frac{q\tau}{m^*}\vec{\upsilon} \times \vec{B}$$

(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

$$\boxed{\vec{\upsilon} \approx -\frac{q\tau}{m^*}\vec{\mathcal{E}}}$$

$$\vec{\upsilon} \approx -\frac{q\tau}{m^*}\vec{\mathcal{E}} + \frac{q^2\tau^2}{\left(m^*\right)^2}\vec{\mathcal{E}} \times \vec{B}$$

(Low B-fields)

B-field dependent DD equation

$$\vec{\upsilon} \approx -\frac{q\tau}{m^*}\vec{\mathcal{E}} + \frac{q^2\tau^2}{\left(m^*\right)^2}\vec{\mathcal{E}} \times \vec{B}$$

$$\langle\vec{\upsilon}\rangle \approx -\frac{q\langle\tau\rangle}{m^*}\vec{\mathcal{E}} + \frac{q^2\langle\tau^2\rangle}{\left(m^*\right)^2}\vec{\mathcal{E}} \times \vec{B}$$

$$\vec{J}_n = -nq\langle\vec{\upsilon}\rangle = nq\left(\frac{q\langle\tau\rangle}{m^*}\right)\vec{\mathcal{E}} - nq\frac{q^2\langle\tau^2\rangle}{\left(m^*\right)^2}\vec{\mathcal{E}} \times \vec{B}$$

$$\vec{J}_n = nq\mu_n\vec{\mathcal{E}} - nq\frac{q\langle\tau\rangle}{m^*}\frac{q\langle\tau\rangle}{m^*}\left(\frac{\langle\tau^2\rangle}{\langle\tau\rangle^2}\right)\vec{\mathcal{E}} \times \vec{B}$$

$$\mu_H = r_H\mu_n$$

$$r_H = \frac{\langle\tau^2\rangle}{\langle\tau\rangle^2}$$

$$\vec{J}_n = nq\mu_n\vec{\mathcal{E}} - \left(\sigma_n\mu_n r_H\right)\vec{\mathcal{E}} \times \vec{B}$$

$$J_i = nq\mu_n\mathcal{E}_i - \sigma_n\mu_H\,\varepsilon_{ijk}\mathcal{E}_j B_k$$

comments

$$\vec{J}_n = nq\mu_n\vec{\mathcal{E}} - \left(\sigma_n\mu_n r_H\right)\vec{\mathcal{E}} \times \vec{B}$$

$$\mu_H = r_H \mu_n$$ "Hall mobility"

$$r_H = \frac{\left\langle \tau^2 \right\rangle}{\left\langle \tau \right\rangle^2}$$ "Hall factor"

(The BTE tells us how to properly compute the averages.)

Lecture 16

$$r_H = \left\langle \tau_f^2 \right\rangle \Big/ \left\langle \tau_f \right\rangle^2$$

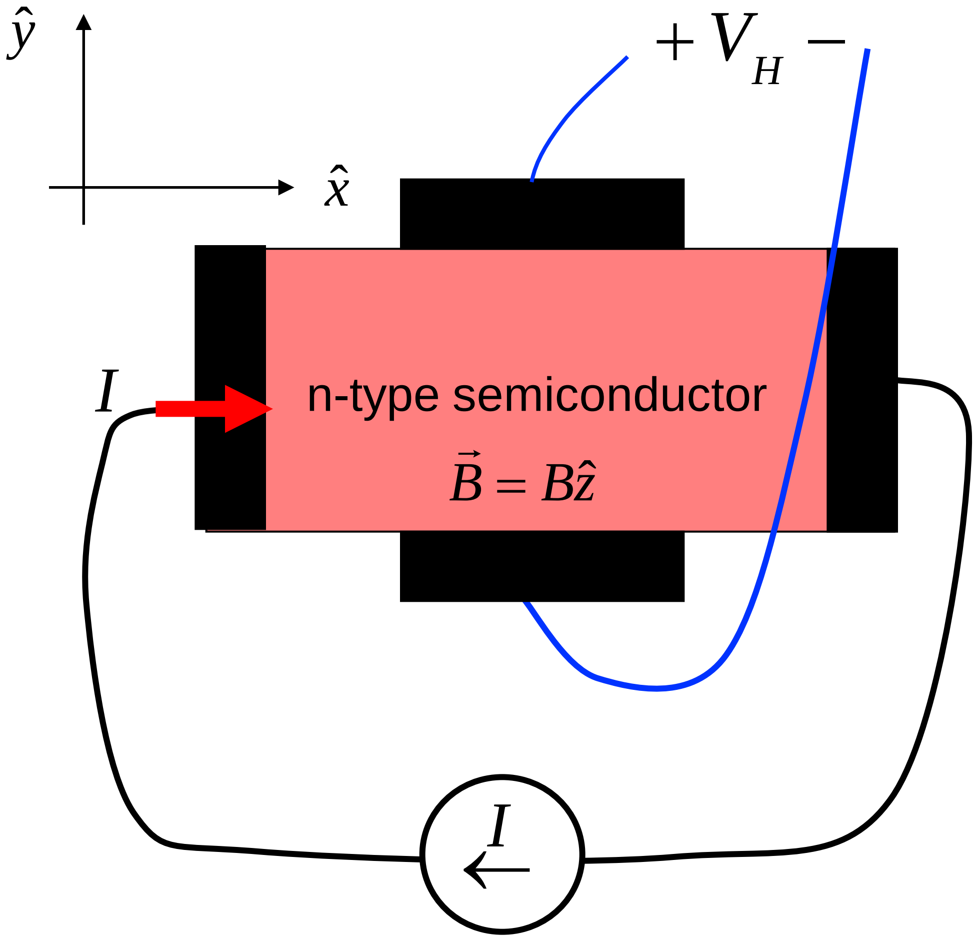
power law scattering:

$$r_H = \frac{\Gamma\left(2s + 5/2\right)\Gamma\left(5/2\right)}{\Gamma\left(s + 5/2\right)^2}$$

ADP scattering: $s = -1/2$
II scattering: $s = 3/2$

$$1.18 < r_H < 1.93$$

Hall effect measurements



current in *x*-direction:

I_x

B-field in *z*-direction:

$$\vec{B} = B\hat{z}$$

Hall voltage measured in the *y*-direction:

$$V_H > 0 \quad (\text{n-type})$$

The Hall effect was discovered by Edwin Hall in 1879 and is widely used to characterize electronic materials. It also finds use magnetic field sensors.

example: Hall effect

$$\vec{J}_n = nq\mu_n\vec{\mathcal{E}} - (\sigma_n\mu_n r_H)\vec{E}\times\vec{B}$$

current in *x*-direction
measure voltage (electric field) in *y*-direction)
B-field in *z*-direction

$$J_x = nq\mu_n\mathcal{E}_x - (\sigma_n\mu_n r_H)\mathcal{E}_y B_z \approx nq\mu_n\mathcal{E}_x$$

$$J_y = 0 = nq\mu_n\mathcal{E}_y + (\sigma_n\mu_n r_H)\mathcal{E}_x B_z$$

$$\mathcal{E}_y = -\mu_n r_H B_z \mathcal{E}_x = -\frac{r_H B_z J_x}{nq}$$

$$\frac{\mathcal{E}_y}{J_x B_z} \equiv R_H = \frac{r_H}{(-q)n}$$

(R_H: Hall coefficient)

"measures" the carrier density

(Hall factor, $1 < r_H < 2$)

example: some numbers

assume “silicon”

$$n_0 = 10^{16} \quad \text{cm}^{-3}$$

$$J_n = 10^2 \quad \text{A/cm}^2$$

$$\mu_n = 1000 \quad \text{cm}^2/\text{V-s}$$

$$r_H = 1$$

$$B_z = 2{,}000 \quad \text{Gauss}$$

$$\left(10^4 \ \text{Gauss} = 1\text{T}\right)$$

$$W_y = 1\ \mu\text{m}$$

$$\mathcal{E}_y = -\mu_n r_H B_z \mathcal{E}_x = -\frac{r_H B_z J_x}{n_0 q}$$

$$\mathcal{E}_y = -1.25 \times 10^2 \ \text{V/cm}$$

$$V_H = -\mathcal{E}_y W_y \approx 13\,\text{mV}$$

outline

1) Magnetoconductivity tensor

2) Resistivity tensor

3) Strong B-fields: Landau levels

4) Shubnikov-DeHaas Oscillations and QHE

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B-field dependent DD equation: again

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = \left(-q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}\right)\tau = m^*\vec{\upsilon}$$

$$\vec{\upsilon} = -\frac{q\tau}{m^*}\vec{\mathcal{E}} - \frac{q\tau}{m^*}\vec{\upsilon} \times \vec{B}$$

(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x - \frac{q\tau}{m^*}\upsilon_y B_z$$

$$\upsilon_y = -\frac{q\tau}{m^*}\mathcal{E}_y + \frac{q\tau}{m^*}\upsilon_x B_z$$

2D problem

z-directed B=field

solution

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x - \frac{q\tau}{m^*}\upsilon_y B_z$$

$$\upsilon_y = -\frac{q\tau}{m^*}\mathcal{E}_y + \frac{q\tau}{m^*}\upsilon_x B_z$$

$$\upsilon_x = -\frac{q\tau}{m^*}\mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z - \left(\frac{q\tau}{m^*}\right)^2 \upsilon_x B_z^2$$

$$\upsilon_x\left(1 + \left(\frac{qB_z}{m^*}\right)^2 \tau^2\right) = -\frac{q\tau}{m^*}\mathcal{E}_x + \left(\frac{q\tau}{m^*}\right)^2 \mathcal{E}_y B_z$$

$$\omega_c = \frac{qB_z}{m^*}$$

“cyclotron frequency”

$$\upsilon_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1 + (\omega_c \tau)^2}$$

solution

$$\upsilon_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1+(\omega_c \tau)^2}$$

Assume *T* = 0K, so there is no need to average the scattering time.

$$J_x = -nq\upsilon_x = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_x - \mu_n B_z \mathcal{E}_y\right)$$

$$J_y = -nq\upsilon_y = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_x + \mu_n B_z \mathcal{E}_y\right)$$

$$\mu_n B_z = \frac{q\tau}{m^*} B_z = \omega_c \tau$$

magneto-conductivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\left(\omega_c \tau = \mu_n B_z\right)$$

$$J_i = \sigma_{ij}\left(B_z\right)\mathcal{E}_j$$

comments

$$\sigma_{ij}(B_z) = \frac{nq\mu_n}{1+\mu_n B_z}\begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}$$

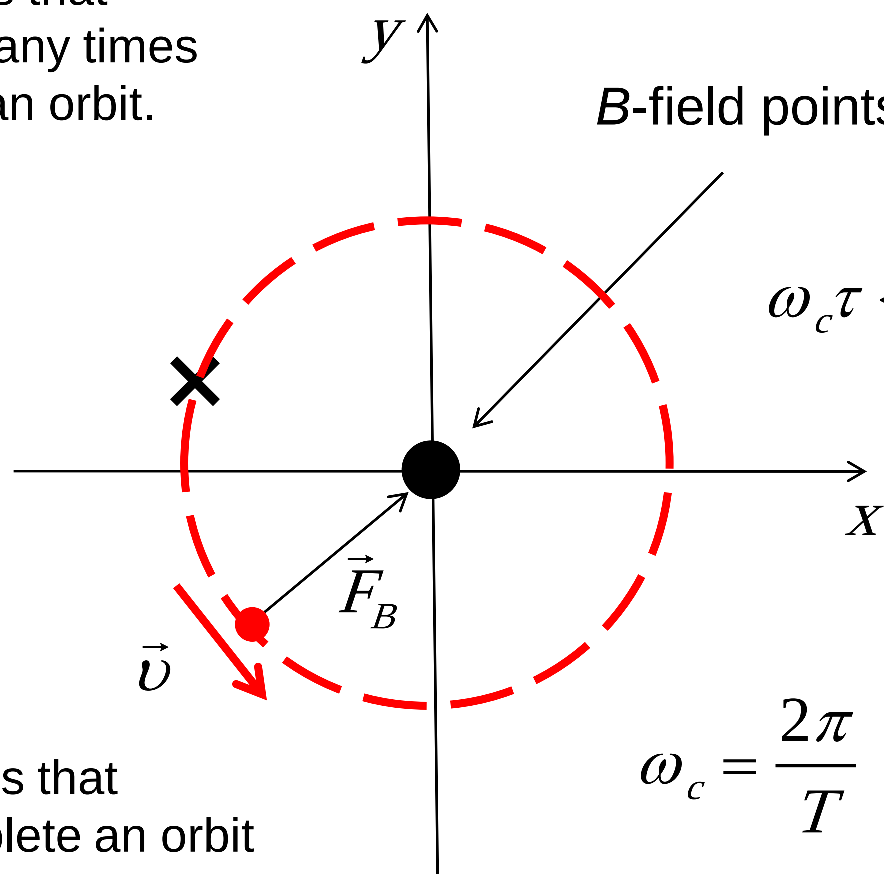
1) A magnetic field affects both the diagonal and off-diagonal components of the magneto-conductivity tensor.

2) Small magnetic field means: $\mu_n B_z << 1$

$$\omega_c \tau << 1$$

more comments

"Low B-field" means that electrons scatter many times before completing an orbit.



$$\omega_c \tau << 1 \rightarrow T >> \tau$$

$$\omega_c = \frac{2\pi}{T}$$

"High B-field" means that electrons can complete an orbit without scattering.

example: some numbers

assume "silicon"

$$n_0 = 10^{16} \text{ cm}^{-3}$$

$$J_n = 10^2 \text{ A/cm}^2$$

$$\mu_n = 1000 \text{ cm}^2/\text{V-s}$$

$$r_H = 1$$

$$B_z = 2,000 \text{ Gauss}$$

$$\left(10^4 \text{ Gauss} = 1\text{T}\right)$$

$$W_y = 1\ \mu\text{m}$$

$$\mu_H B_z \approx 0.02 << 1$$

$$\mu_H B_z \approx 10 \rightarrow B_z = 100\ T$$

Hall effect measurements with typical laboratory-sized magnets are in the low B-field regime. Except – for very high mobility sample such as modulation doped films.)

Birck Nanotechnology Center: 1-8 T

National High Magnetic Field Lab (Florida State Univ.): 45 T

magneto-resistivity tensor

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$J_i = \sigma_{ij}\left(B_z\right)\mathcal{E}_j$$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{bmatrix} \sigma_L & -\sigma_T \\ \sigma_T & \sigma_L \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \begin{bmatrix} \rho_L & \rho_T \\ -\rho_T & \rho_L \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

$$\rho_L = \frac{\sigma_L}{\sigma_L^2 + \sigma_T^2} = \frac{1}{\sigma_n}$$

$$\rho_T = \frac{\sigma_T}{\sigma_L^2 + \sigma_T^2} = \frac{\mu_n B_z}{\sigma_n}$$

comparison

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

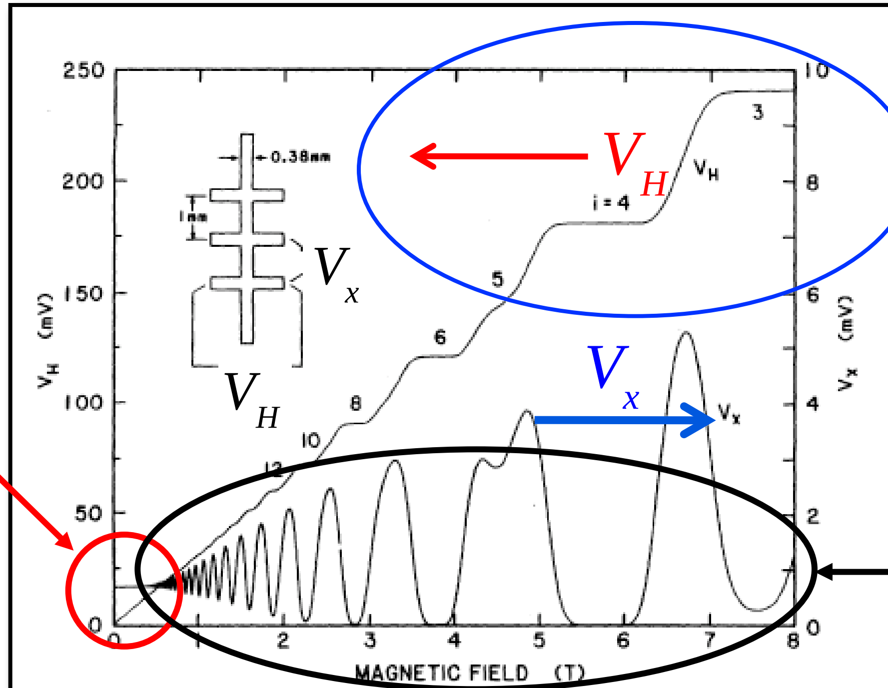
$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \frac{1}{\sigma_n} \begin{bmatrix} 1 & \mu_n B_z \\ -\mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

1) Longitudinal magneto-resistance is independent of *B*
2) Hall voltage is proportional to *B*

Hall and integer quantum Hall effect

Hall effect

$V_H \sim B$
V_x independent of B



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Quantum Hall effect

Longitudinal magneto-resistance

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outline

1) Magnetoconductivity tensor

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3) **Strong B-fields: Landau levels**

4) Shubnikov-DeHaas Oscillations and QHE

5) Summary

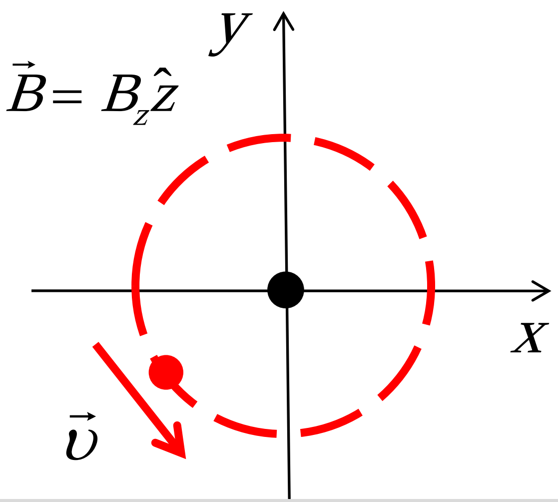
Schrödinger equation with a B-field

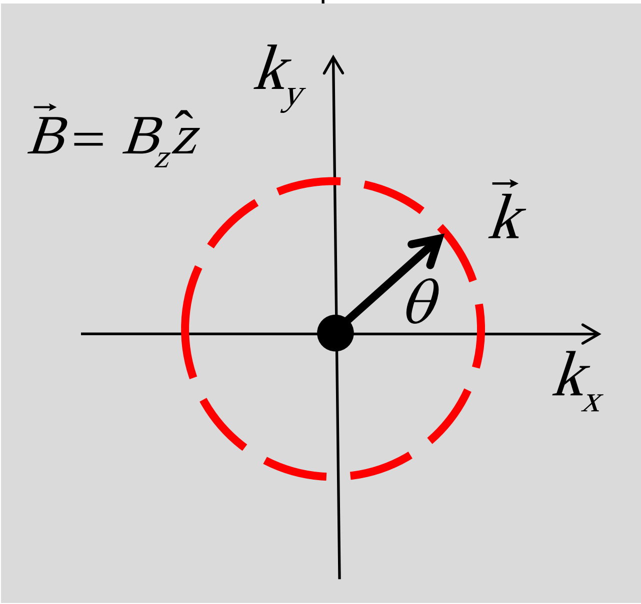
$$\left[\varepsilon_1 + \frac{\left(i\hbar\nabla + q\vec{A}\right)}{2m^*} + U\left(x, y\right)\right]\Psi\left(x, y\right) = E\Psi\left(x, y\right)$$

$$\vec{B} = \nabla \times \vec{A}$$

See S. Datta, *Electronic Transport in Mesoscopic Systems*, Cambridge, 1995, pp. 29-27.

cyclotron frequency





$$\frac{d\left(\hbar\vec{k}\right)}{dt} = -q\vec{\upsilon}\times\vec{B}$$

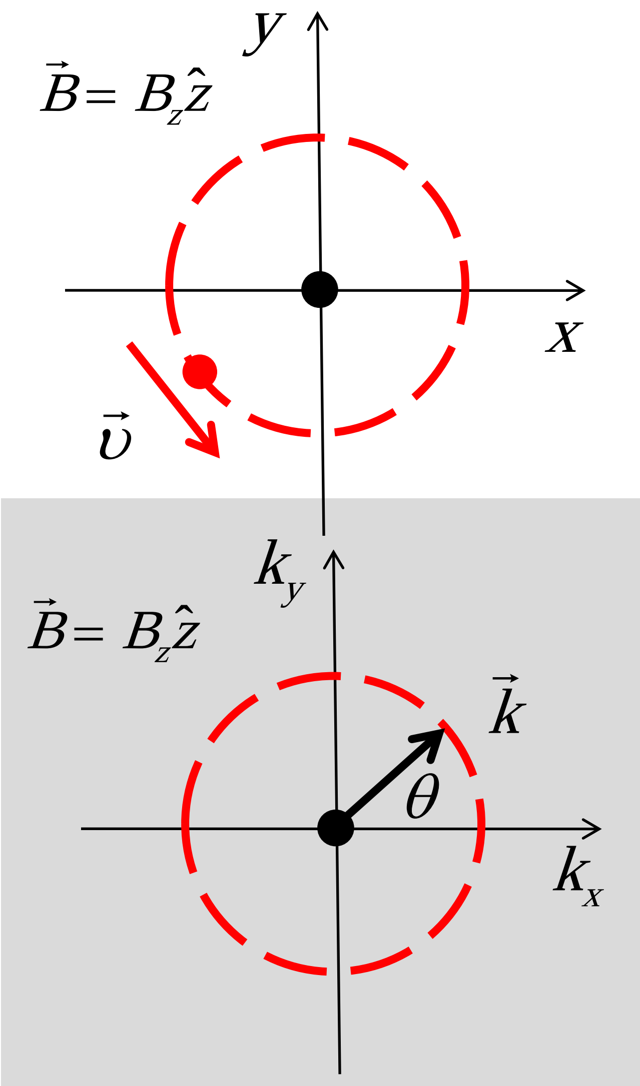
$$\hbar\frac{dk_x}{dt} = -q\upsilon_y B_z = \hbar k\frac{d\cos\theta}{dt} = -q\upsilon\sin\theta B_z$$

$$\hbar\frac{dk_y}{dt} = +q\upsilon_x B_z = \hbar k\frac{d\sin\theta}{dt} = +q\upsilon\cos\theta B_z$$

$$\frac{d^2\cos\theta}{dt^2} = -\left(\frac{q\upsilon B_z}{\hbar k}\right)^2\cos\theta = -\omega_c^2\cos\theta$$

$$\cos\theta\left(t\right) = \cos\theta\left(0\right)e^{i\omega_c t}$$

cyclotron frequency



$$\frac{d\left(\hbar\vec{k}\right)}{dt} = -q\vec{\upsilon}\times\vec{B}$$

harmonic oscillator:

$$\omega_c = \left(\frac{q\upsilon B_z}{\hbar k}\right)$$

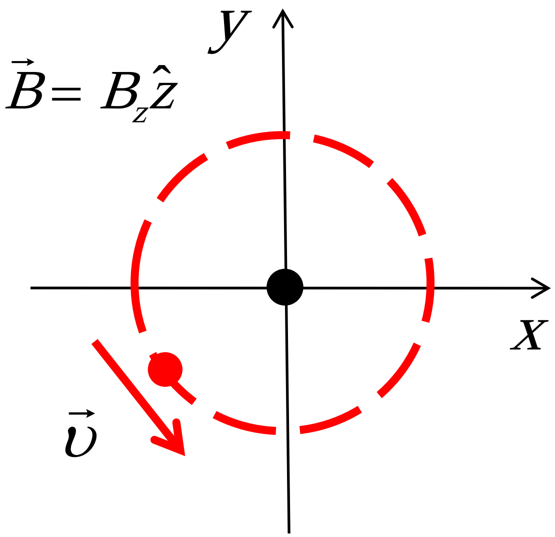
Quantum mechanically:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

"Landau levels"

cyclotron frequency

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c \qquad \omega_c = \left(\frac{q\upsilon B_z}{\hbar k}\right)$$



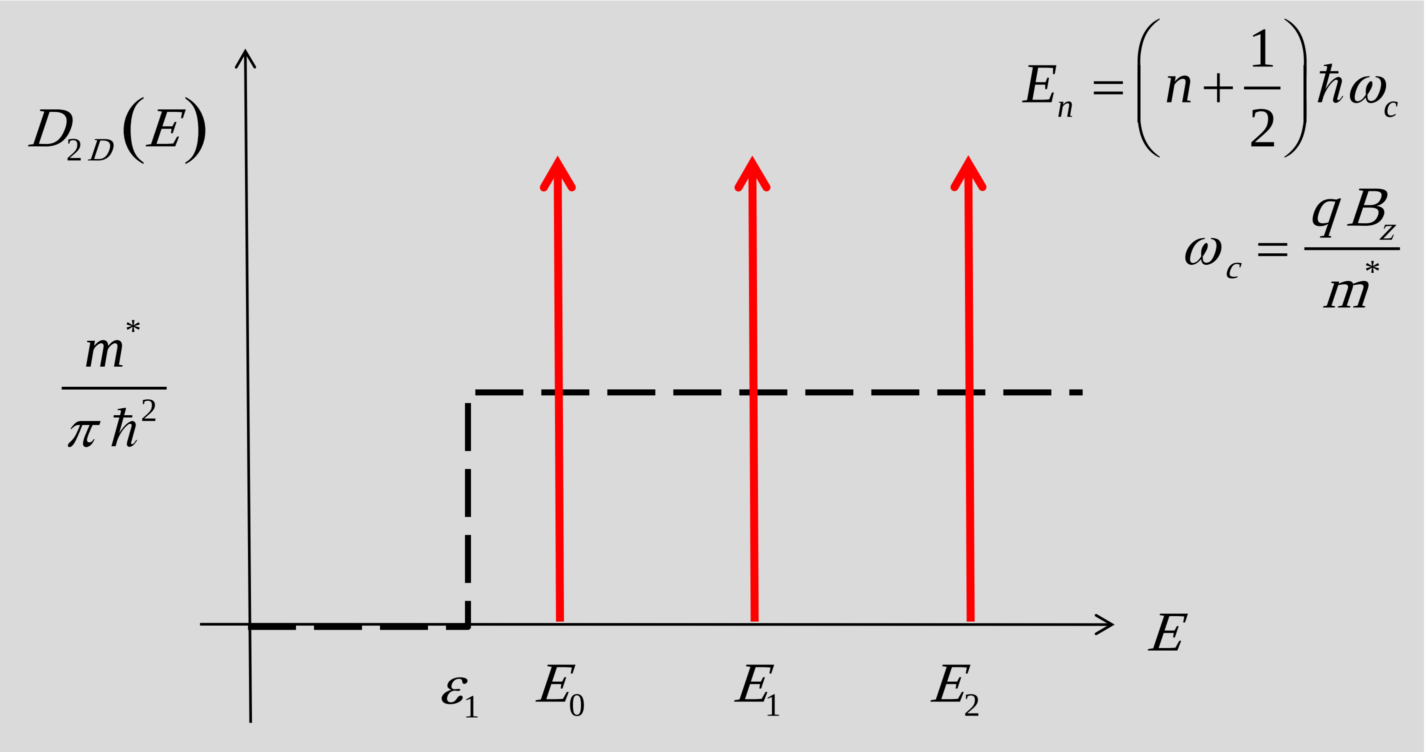
i) parabolic energy bands:

$$\upsilon = \hbar k / m^* \qquad \omega_c = \frac{qB_z}{m^*}$$

ii) graphene:

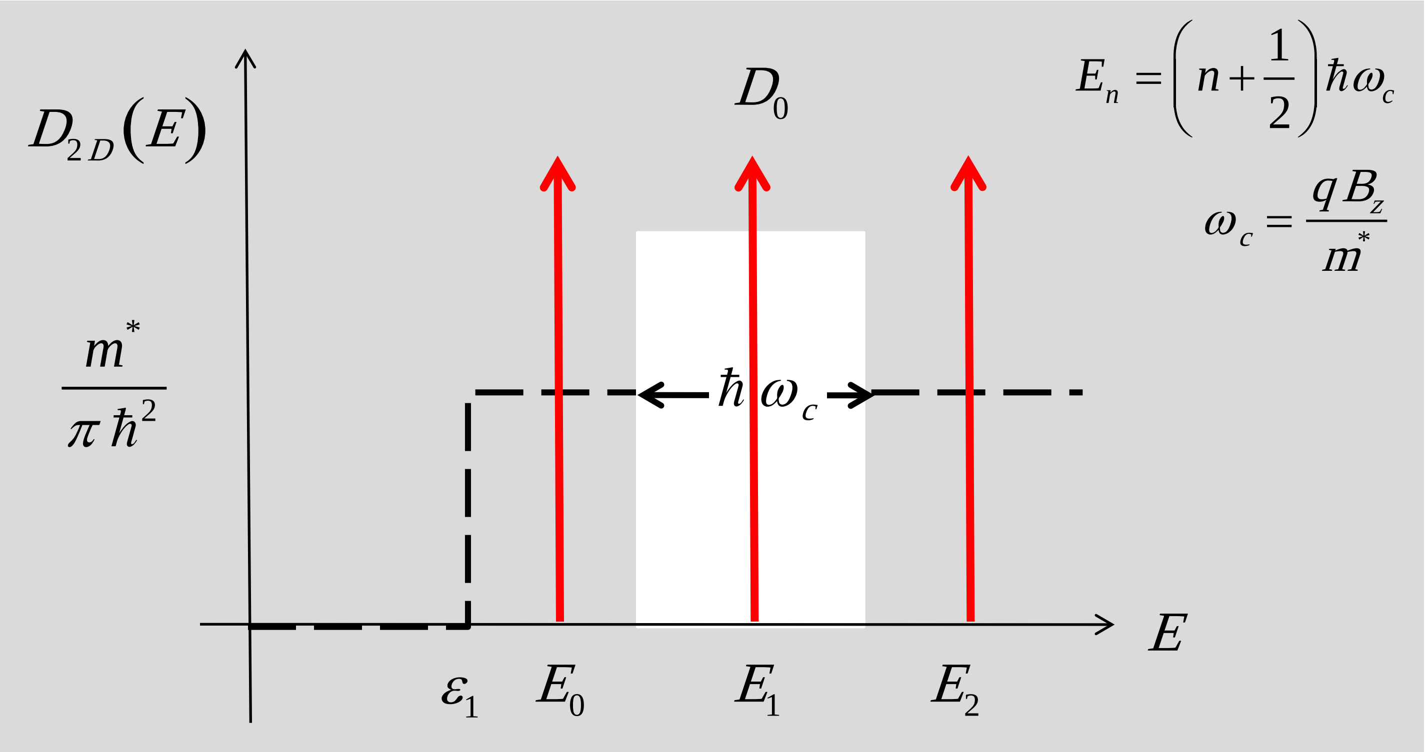
$$E = \hbar \upsilon_F k \qquad \omega_c = \frac{qB_z}{\left(E / \upsilon_F^2\right)}$$

effect on DOS



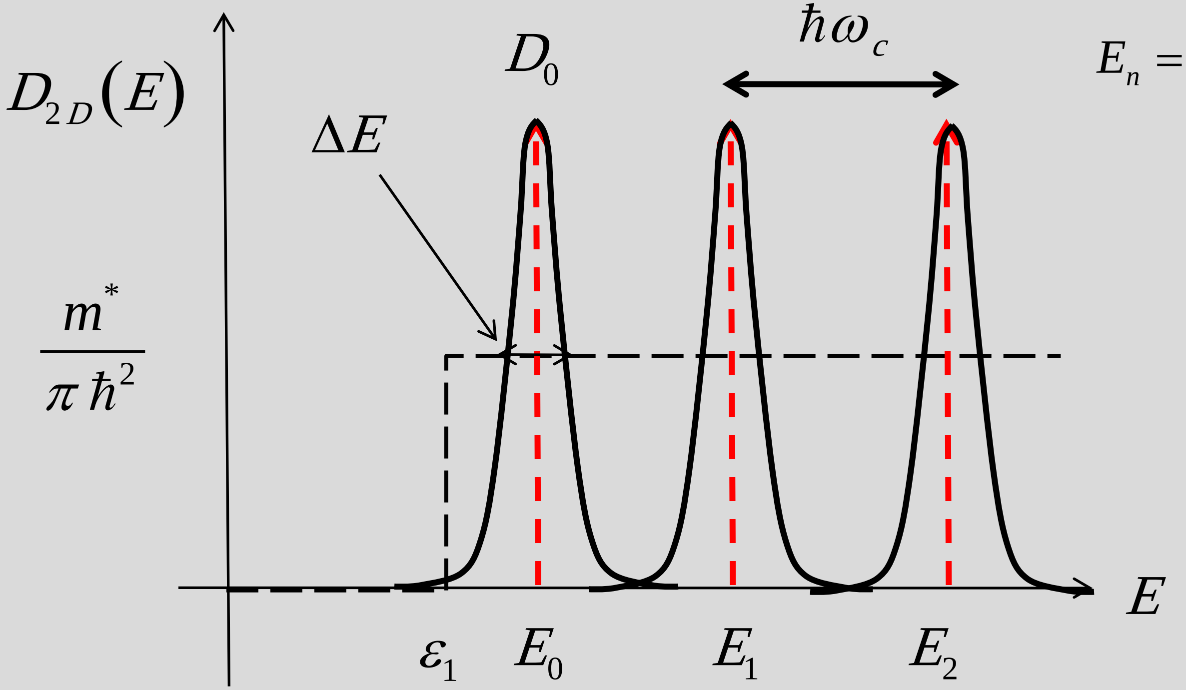
$$D_{2D}(E, B_z) = D_0 \sum_{n=0}^{\infty} \delta\left[E - \varepsilon_1 - \left(n + \frac{1}{2}\right)\hbar\omega_c\right]$$

degeneracy of Landau levels



$$D_0 = \hbar\omega_c \times \frac{m^*}{\pi\hbar^2} = \frac{2qB_z}{h}$$

broadening



$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

$$\omega_c = \frac{qB_z}{m^*}$$

$$D_0 = \frac{2qB_z}{h}$$

$$\Delta E \Delta t = \hbar$$

$$\Delta E \approx \hbar/\tau$$

to observe Landau levels: $\hbar\omega_c >> \Delta E \rightarrow \omega_c\tau >> 1$

example

If B = 1T, how many states are there in each LL?

$$D_0 = \frac{2qB_z}{h} = 4.8 \times 10^{10} \text{ cm}^{-2}$$

If $n_S = 5 \times 10^{11}$ cm^{-2}, then 10.4 LL's are occupied.

How high would the mobility need to be to observe these LL's?

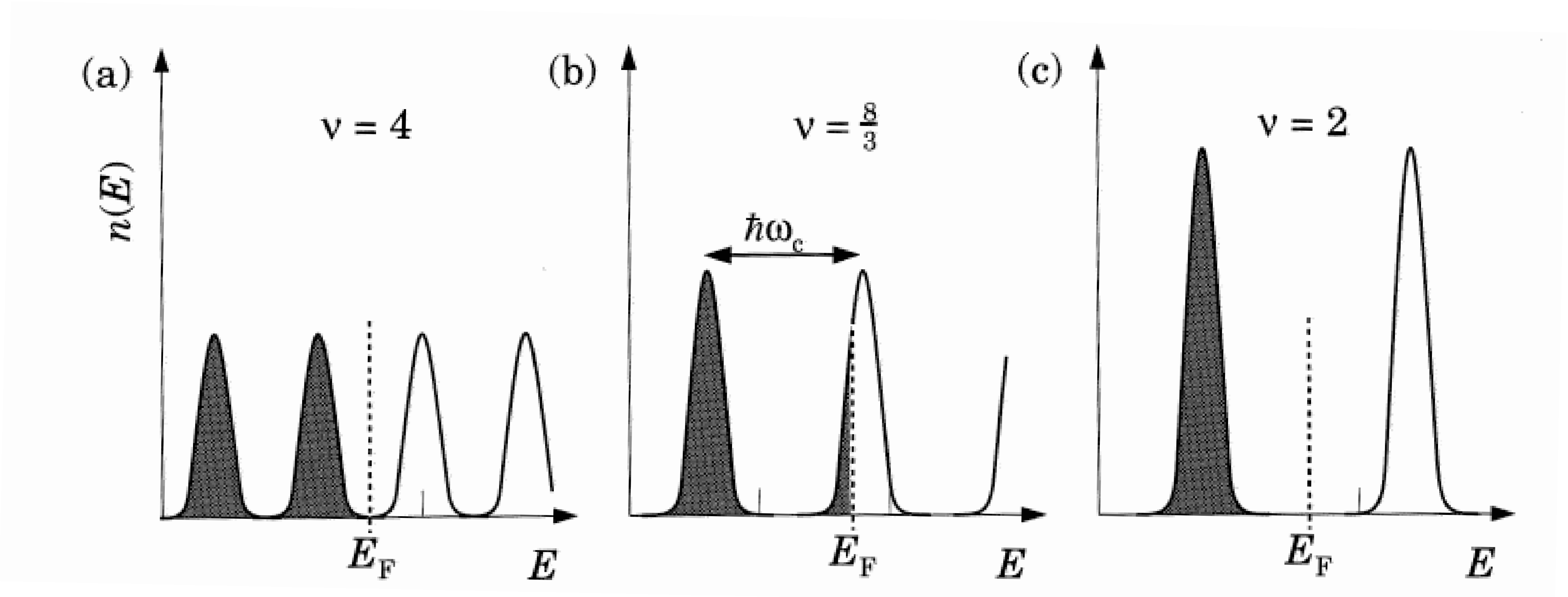
$$\mu B > 1 \rightarrow \mu > 10{,}000 \text{ cm}^2/\text{V-s} \quad (B = 1\text{ T})$$

"modulation-doped semiconductors"

variation of LL's with *B*

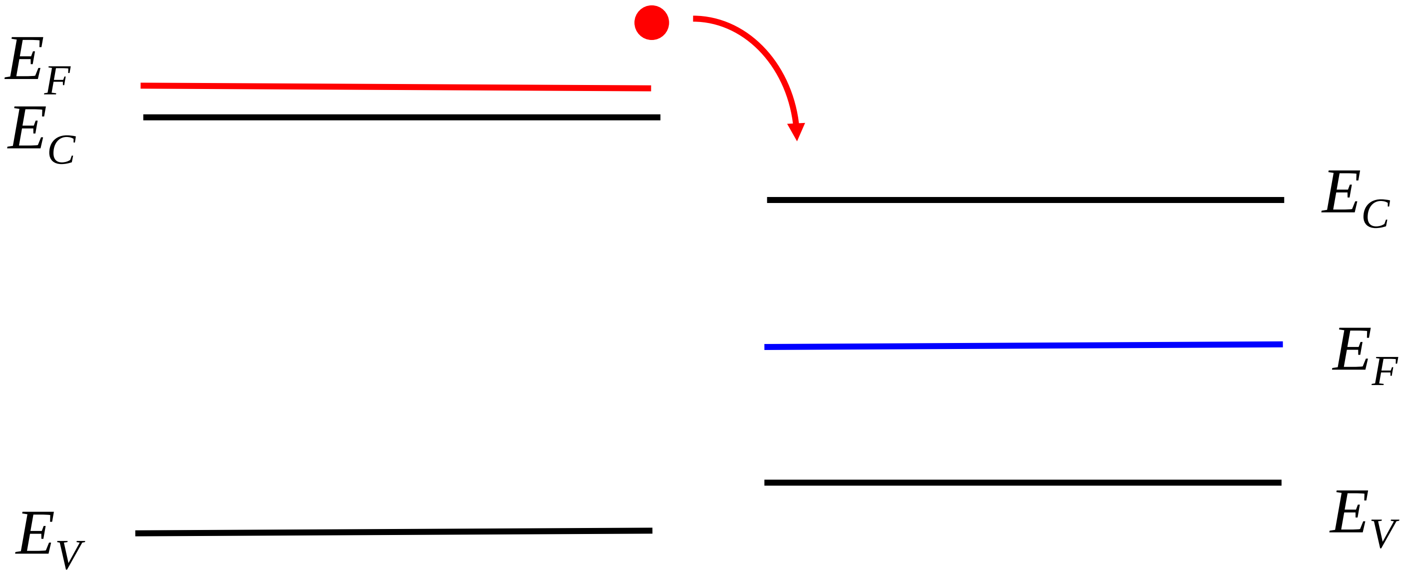
1) degeneracy of each level increases with *B* $D_0 = 2qB_z/h$

2) spacing of levels increases $\omega_c = qB_z/m^*$



(from J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998, Fig. 6.8, p. 226)

modulation doping



wide band gap
doped n$^+$

small band gap
nominally undoped

R. Dingle, et al, *Appl. Phys. Lett.*, **33**, 665, 1978.

outline

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2) Resistivity tensor

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4**) Shubnikov-DeHaas Oscillations and QHE**

5) Summary

SdH oscillations

Longitudinal magneto-resistance

"Shubnikov-deHaas (SdH) oscillations"

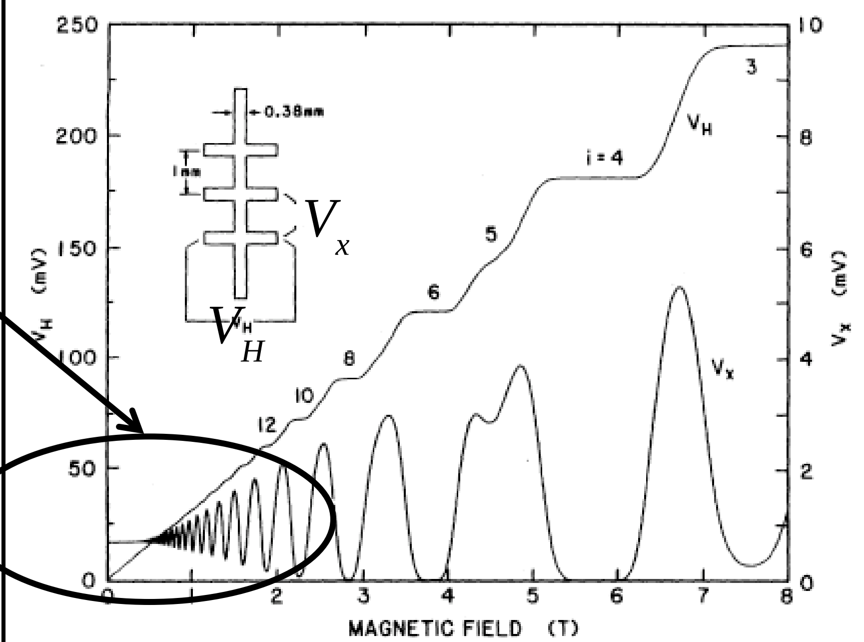

Fig. 1. Recording of V_H and V_x versus magnetic field for a GaAs device cooled to 1.2 K. The current is 25.5 μA.

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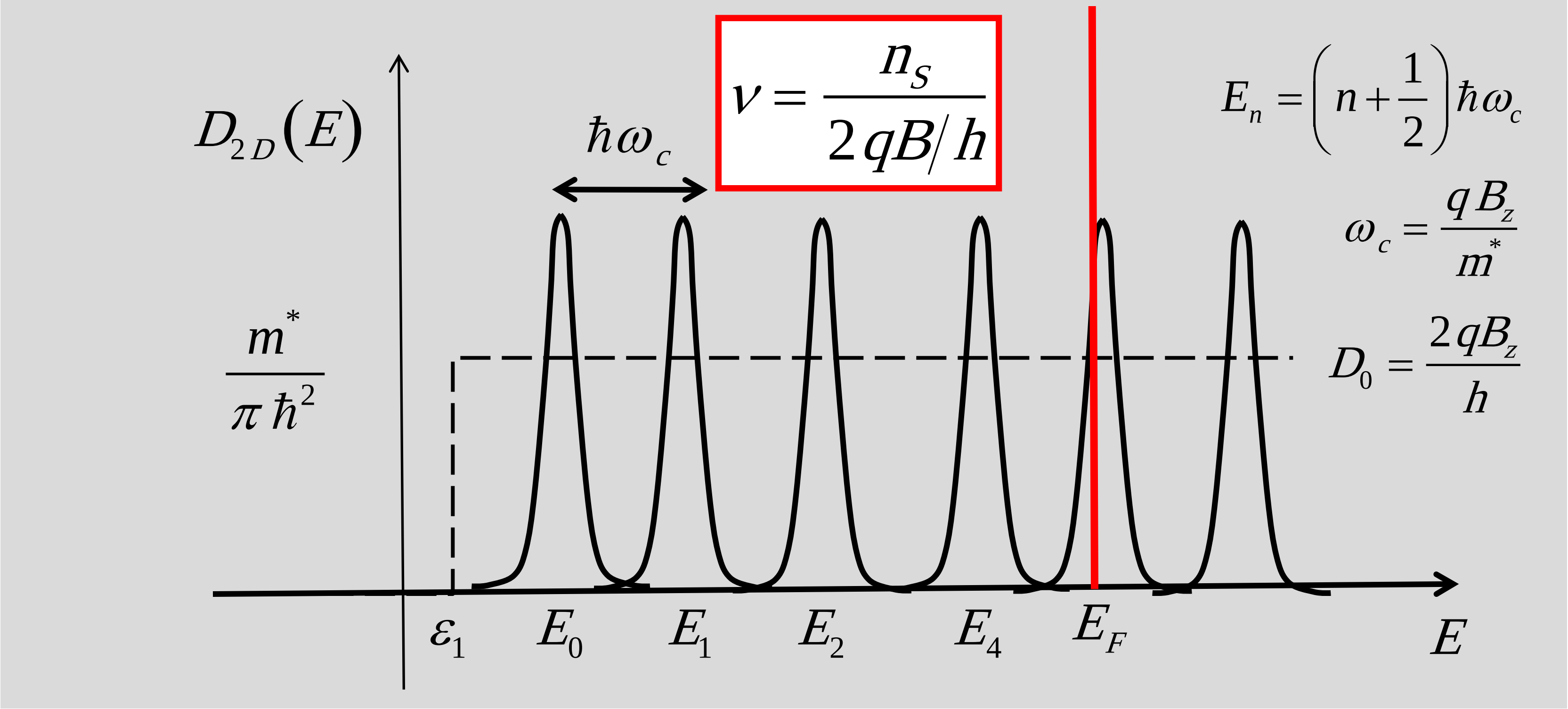
LL filling



For a given sheet carrier density, n_s, some (fractional) number of LL's are occupied.

SdH oscillations

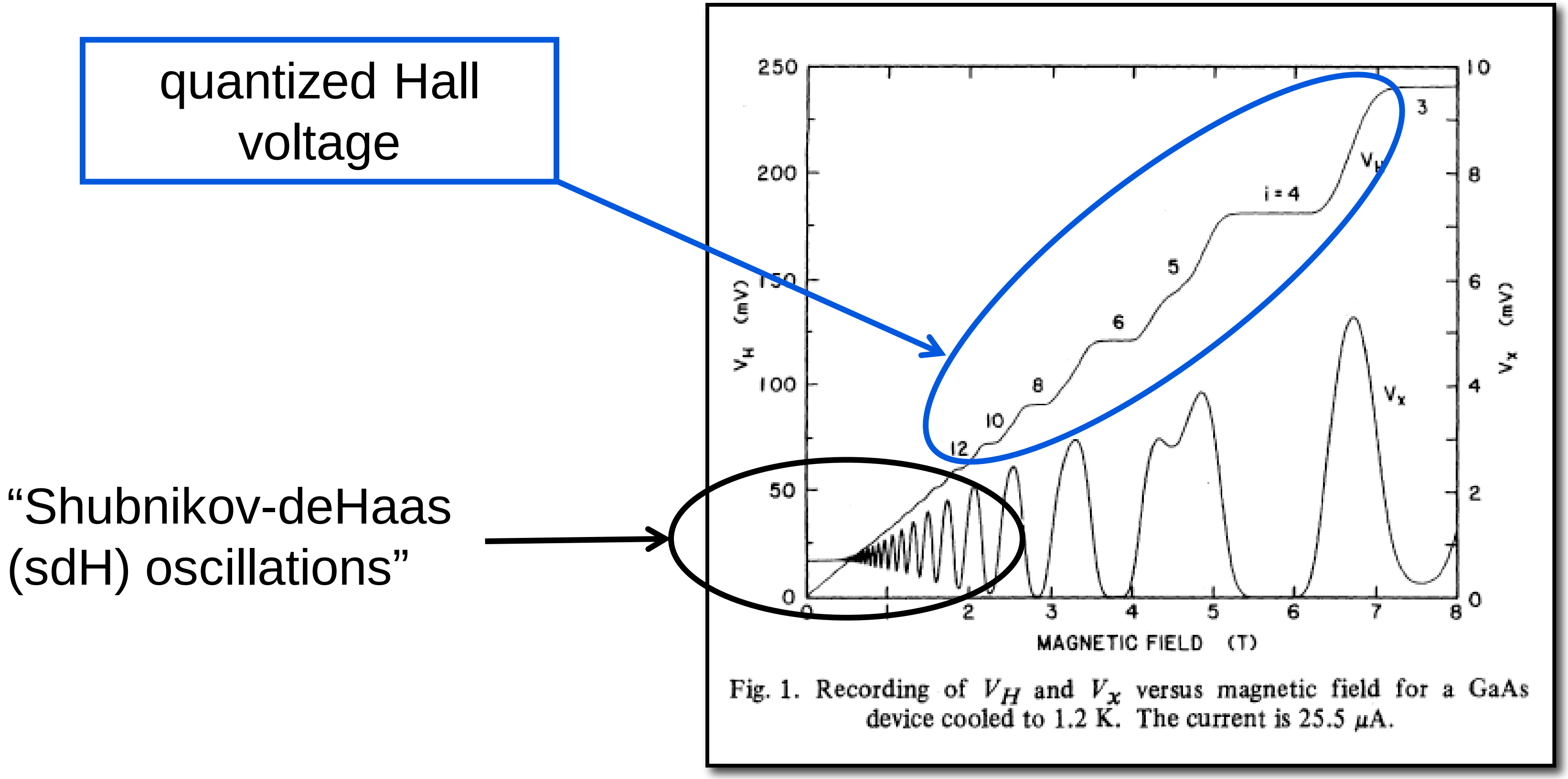
As the *B*-field is varied, the longitudinal resistance will oscillate as the number of filled LL's changes. The period of the oscillation corresponds to the change in filling factor from one integer to the next.

$$\nu_1 - \nu_2 = 1$$

$$\frac{n_S}{2qB_1/h} - \frac{n_S}{2qB_2/h} = 1$$

$$n_S = \frac{2q}{h}\left(\frac{1}{1/B_1 - 1/B_2}\right)$$

SdH oscillations



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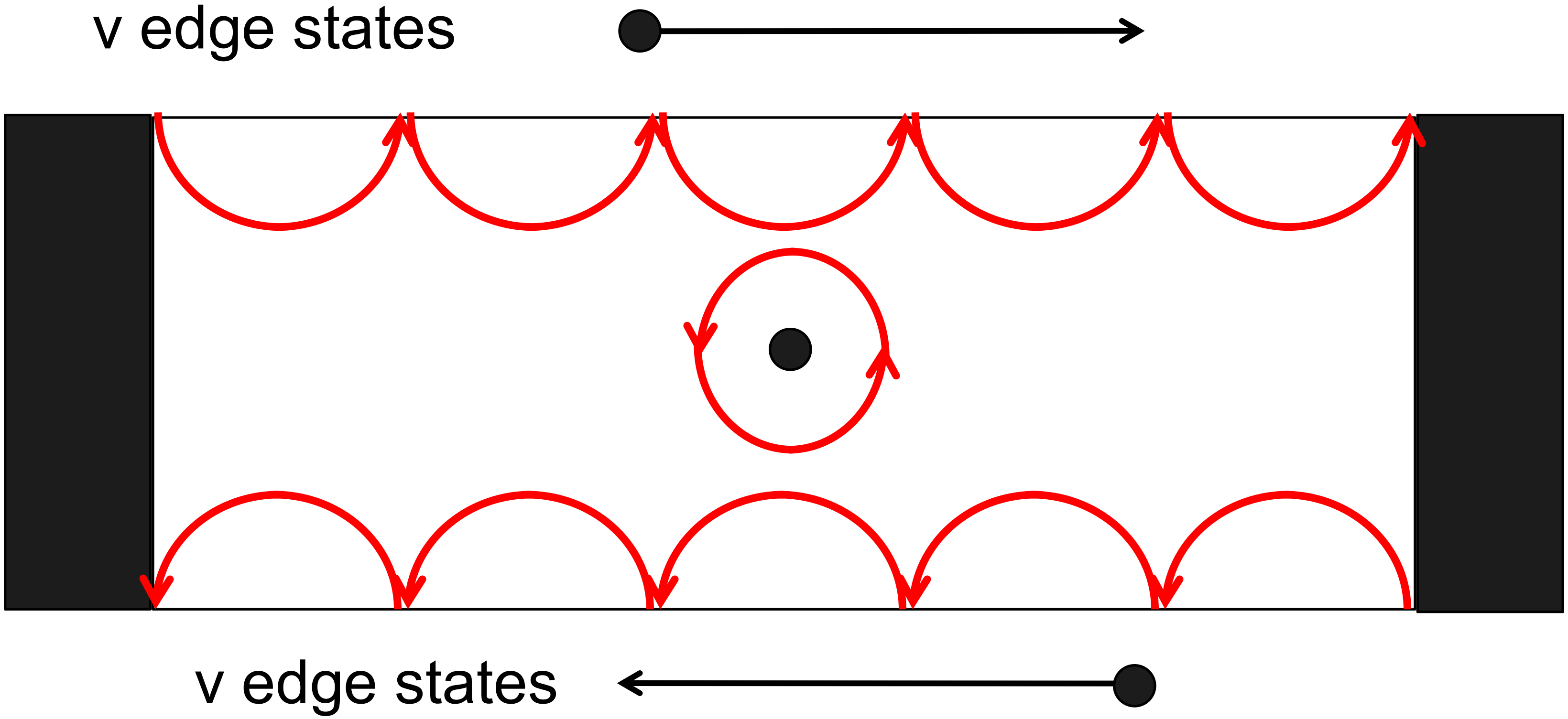
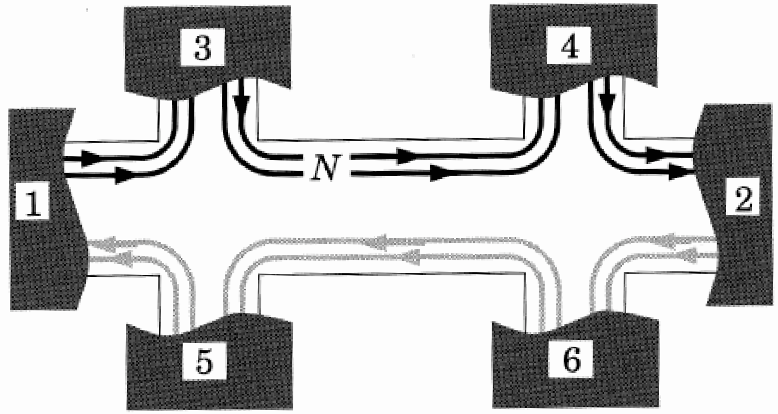
edge states



Assume that the Fermi level lies between Landau levels in the bulk, then only the edge states are occupied (edge states play the role of modes).

QHE: quantized Hall Voltage

T = 1 because +*v* and –*v* channels are spatially isolated.



(from J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998, Fig. 6.18, p. 240)

$V_1 = 0$ and $V_2 = +V$

$V_3 = V_4 = V_1 = 0$

$$I = \frac{2q^2}{h} \nu V$$

$V_6 = V_5 = V_2 = +V$

$$V_H = V_3 - V_5 = \frac{1}{\nu} \frac{h}{2q^2} I$$

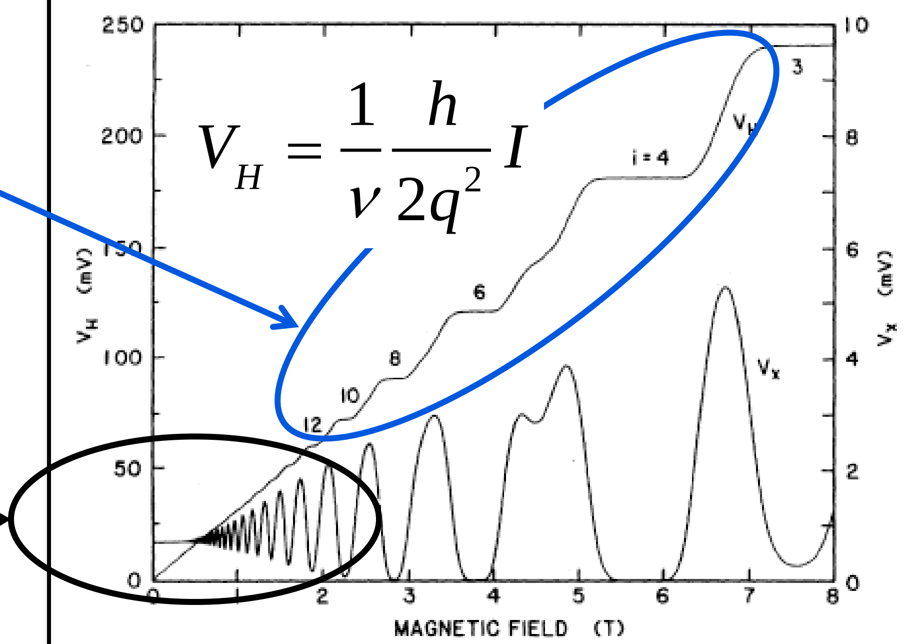
"quantized Hall resistance"

SdH oscillations

quantized Hall voltage

$$V_H = \frac{1}{\nu}\frac{h}{2q^2} I$$

"Shubnikov-deHaas (sdH) oscillations"



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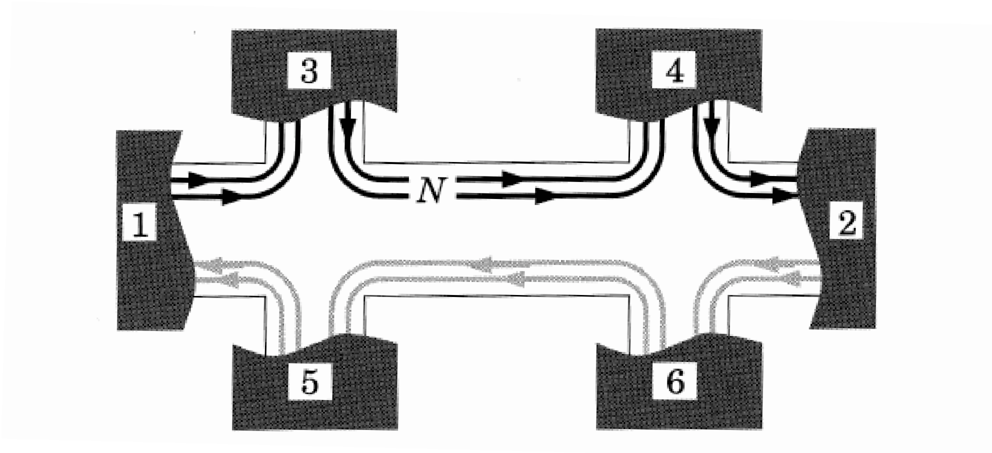
QHE: zero longitudinal magnetoresistance



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$V_1 = 0$ and $V_2 = +V$

$V_3 = V_4 = V_1 = 0$

$$I = \frac{2q^2}{h}\nu V$$

$V_6 = V_5 = V_2 = +V$

$$V_x = V_4 - V_3 = 0$$

zero longitudinal resistance

for a more complete discussion

See Chapter 4 in *Electronic Transport in Mesoscopic Systems*, Supriyo Datta, Cambridge, 1995.

J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998.

D. F. Holcomb, “Quantum electrical transport in samples of limited dimensions,” *American J. Physics*, **64**, pp. 278-297, 1999.