ECE-656: Fall 2009

Lecture 19: Characteristic Times

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the BTE

$$f\left(\vec{r},\vec{p},t\right): \qquad \frac{\partial f}{\partial t}+\vec{\upsilon}\bullet\nabla_r f+\vec{F}_e\bullet\nabla_p f=\hat{C}f$$

$$\hat{C}f=\sum_{p'}S\left(\vec{p}',\vec{p}\right)f\left(\vec{p}'\right)\left[1-f\left(\vec{p}\right)\right]-\sum_{p'}S\left(\vec{p},\vec{p}'\right)f\left(\vec{p}\right)\left[1-f\left(\vec{p}'\right)\right]$$

1) How do we calculate *S*(*p*, *p*')?

2) How do we obtain physically-relevant, "characteristic times" from *S*(*p*, *p*')?

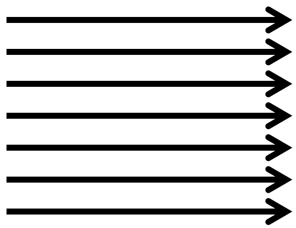
outline

1) **Characteristic times**

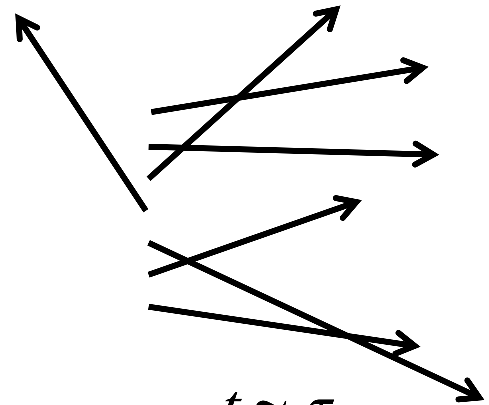
2) Relaxation Time Approximation and τ_f

characteristic times

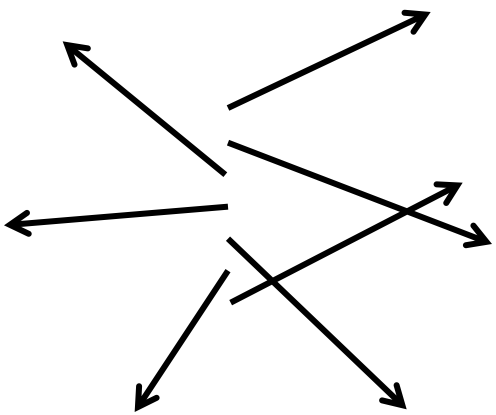
$\vec{p}(t=0)$



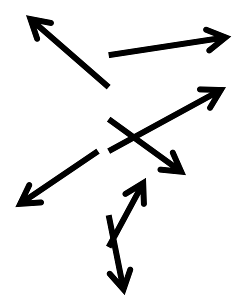
$t=0$



$t \approx \tau$



$t \approx \tau_m \geq \tau$



$t \approx \tau_E > \tau_m \geq \tau$

scattering rate

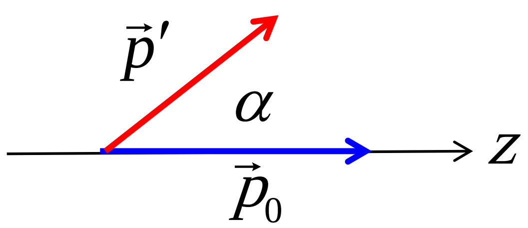
$S(\vec{p}_0, \vec{p}\,')$ transition rate for scattering from p_0 to p'

$$\frac{1}{\tau(\vec{p}_0)} = \sum_{\mathbf{p}',\uparrow} S(\vec{p}_0, \vec{p}\,')\left[1 - f(\vec{p}\,')\right]$$

(no spin flip scattering)

$$\frac{1}{\tau(\vec{p}_0)} = \sum_{\mathbf{p}',\uparrow} S(\vec{p}_0, \vec{p}\,')$$

"out-scattering rate"

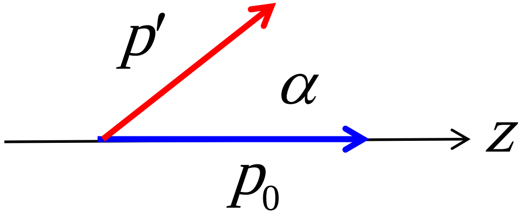


$$\frac{1}{\tau(\vec{p}_0)} = \frac{\Omega}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\alpha \, d\alpha \int_0^{\infty} p'^2 dp' \, S(\vec{p}_0, \vec{p}\,')$$

momentum relaxation time

$$\frac{1}{\tau_m(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') \frac{\Delta p_z}{p_{z0}}$$

$$\frac{\Delta p_z}{p_{z0}} = \frac{p_{z0} - p'_z}{p_{z0}} = \left(1 - p'_z / p_{z0}\right)$$



$$\frac{1}{\tau_m(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') \left[1 - \left(p'/p_0\right)\cos\alpha\right]$$

momentum relaxation time and scattering time

$$\frac{1}{\tau_m(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')\left[1 - (p'/p_0)\cos\alpha\right]$$

$$\frac{1}{\tau_m(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') - \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')(p'/p_0)\cos\alpha$$

$$\frac{1}{\tau_m(\vec{p}_0)} = \frac{1}{\tau(\vec{p}_0)} - \underbrace{\sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')(p'/p_0)\cos\alpha}$$

$$\frac{1}{\tau_m(\vec{p}_0)} = \frac{1}{\tau(\vec{p}_0)}$$

scattering

Zero if *S* contains no angular dependence (isotropic scattering)

energy relaxation time

$$\frac{1}{\tau_E(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') \frac{\Delta E}{E_0}$$

$$\frac{\Delta E}{E_0} = \frac{E(\vec{p}_0) - E(\vec{p}')}{E(\vec{p}_0)}$$

$\Delta E = 0$ (elastic)

$$\frac{1}{\tau_E(\vec{p}_0)} = 0 \quad \tau_E(\vec{p}_0) = \infty$$

$\Delta E = \hbar\omega$ (phonon emisson)

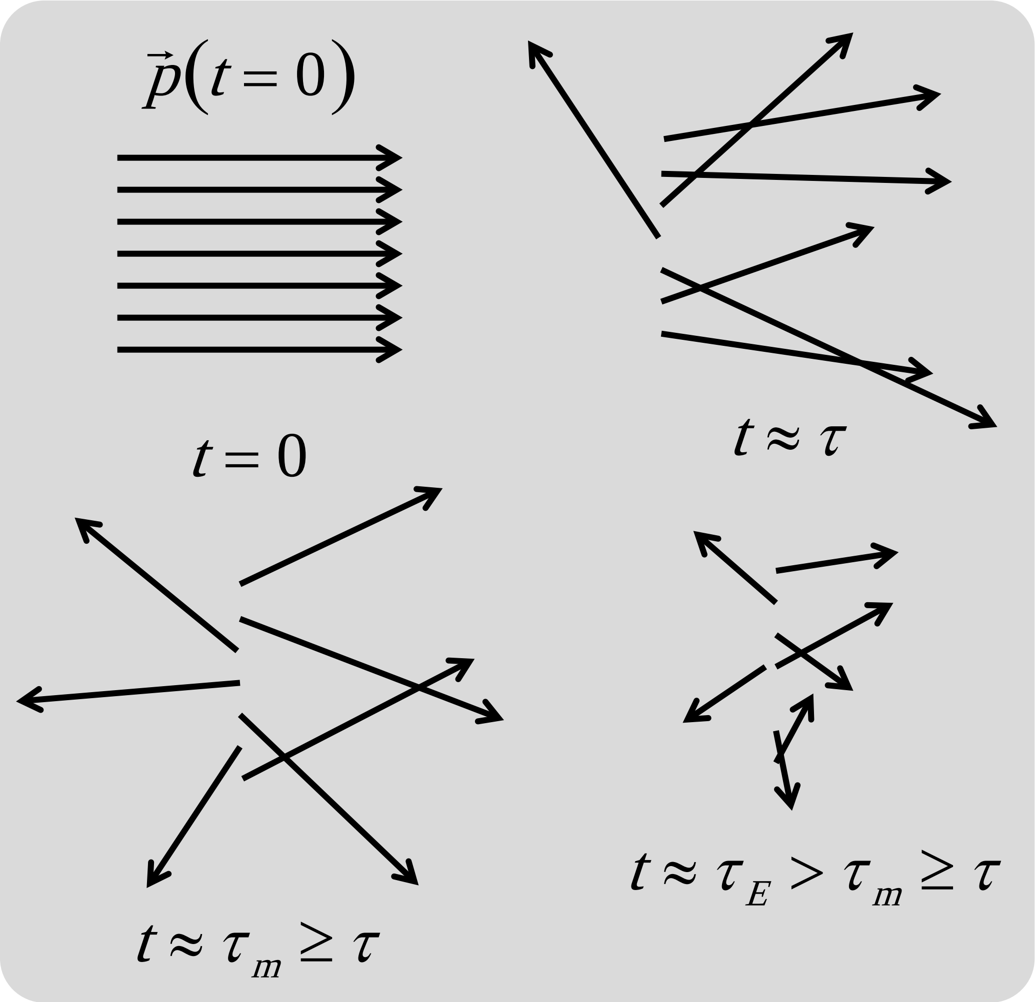
$$\frac{1}{\tau_E(\vec{p}_0)} = \frac{\hbar\omega}{E_0}\frac{1}{\tau} \quad \tau_E(\vec{p}_0) = \frac{E_0}{\hbar\omega}\tau \geq \tau$$

comparison of times

$$\frac{1}{\tau(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')$$

$$\frac{1}{\tau_m(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') \frac{\Delta p_z}{p_z}$$

$$\frac{1}{\tau_E(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}') \frac{\Delta E}{E_0}$$



any relaxation time

$$\frac{1}{\tau_\phi(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')\frac{\Delta\phi}{\phi(\vec{p}_0)}$$

$$\frac{\Delta\phi}{\phi(\vec{p}_0)} = \frac{\phi(\vec{p}_0) - \phi(\vec{p}')}{\phi(\vec{p}_0)}$$

$$= \left[1 - \phi(\vec{p}')/\phi(\vec{p}_0)\right]$$

$$\frac{1}{\tau_\phi(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')\left[1 - \phi(\vec{p}')/\phi(\vec{p}_0)\right]$$

example:

$$\phi(\vec{p}_0) = \upsilon_z(\vec{p}_0) \qquad \frac{1}{\tau_\phi(\vec{p}_0)} = \sum_{\vec{p}',\uparrow} S(\vec{p}_0, \vec{p}')\left[1 - \upsilon_z(\vec{p}')/\upsilon_z(\vec{p}_0)\right]$$

outline

1) Characteristic times

2) Relaxation Time Approximation and τ_f

collision integral and RTA

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}') f(\vec{p})$$ (non-degenerate)

$$\hat{C}f(\vec{r}, \vec{p}, t) = -\frac{f - f_S}{\tau_f(\vec{r}, \vec{p})}$$ (Relaxation Time Approximation)

1) Under what conditions can we approximate the collision integral with the RTA?

2) When we can, how is τ_f defined?

simplified collision integral

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\, f(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f(\vec{p})$$ (non-degenerate)

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\left[f_0(\vec{p}') + f_A(\vec{p}') \right] - \sum_{p'} S(\vec{p}, \vec{p}')\left[f_0(\vec{p}) + f_A(\vec{p}) \right]$$

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p})\, f_0(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f_0(\vec{p}) = 0$$ (equilibrium)

$$\hat{C}f = \sum_{p'} S(\vec{p}', \vec{p}) f_A(\vec{p}') - \sum_{p'} S(\vec{p}, \vec{p}')\, f_A(\vec{p})$$

equilibrium simplification

$$\hat{C}f = \sum_{p'} S(\vec{p}',\vec{p}) f(\vec{p}') - \sum_{p'} S(\vec{p},\vec{p}') f(\vec{p})$$

(return to the first form of the equation)

Can we relate $S(\vec{p}',\vec{p})$ and $S(\vec{p},\vec{p}')$? Yes, in equilibrium.

$$\hat{C}f = \sum_{p'} S(\vec{p}',\vec{p}) f_0(\vec{p}') - \sum_{p'} S(\vec{p},\vec{p}') f_0(\vec{p}) = 0$$

(equilibrium)

$$S(\vec{p}',\vec{p}) f_0(\vec{p}') - S(\vec{p},\vec{p}') f_0(\vec{p}) = 0$$

(detailed balance)

$$S(\vec{p}',\vec{p}) = S(\vec{p},\vec{p}') \frac{f_0(\vec{p})}{f_0(\vec{p}')}$$

Applies out of equilibrium too.

RTA (i)

$$\hat{C}f = \sum_{p'} S(\vec{p}\,',\vec{p}) f_A(\vec{p}\,') - \sum_{p'} S(\vec{p},\vec{p}\,') f_A(\vec{p}) \qquad S(\vec{p}\,',\vec{p}) = S(\vec{p},\vec{p}\,')\frac{f_0(\vec{p})}{f_0(\vec{p}\,')}$$

Continue with our simplified collision integral.

And with the eq. simplification for the transition rate.

$$\hat{C}f = \sum_{p'} S(\vec{p},\vec{p}\,')\frac{f_0(\vec{p})}{f_0(\vec{p}\,')} f_A(\vec{p}\,') - \sum_{p'} S(\vec{p},\vec{p}\,') f_A(\vec{p})$$

$$\hat{C}f(\vec{p}) = -f_A(\vec{p})\sum_{p'} S(\vec{p},\vec{p}\,')\left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}\,')}\frac{f_A(\vec{p}\,')}{f_A(\vec{p})}\right)$$

$$\sum_{p'} S(\vec{p},\vec{p}\,')\left(1 - \frac{f_0(\vec{p})}{f_0(\vec{p}\,')}\frac{f_A(\vec{p}\,')}{f_A(\vec{p})}\right) = \frac{1}{\tau_f(\vec{p})}$$

The characteristic time should be independent of *f*.

RTA (ii)

$$\hat{C}f\left(\vec{p}\right) = -f_A\left(\vec{p}\right)\sum_{p'} S\left(\vec{p},\vec{p}'\right)\left(1 - \frac{f_0\left(\vec{p}\right)}{f_0\left(\vec{p}'\right)}\frac{f_A\left(\vec{p}'\right)}{f_A\left(\vec{p}\right)}\right)$$

i) isotropic scattering: $S\left(\vec{p},\vec{p}'\right) = S\left(\vec{p},-\vec{p}'\right)$

$$\hat{C}f = -f_A\left(\vec{p}\right)\sum_{p'} S\left(\vec{p},\vec{p}'\right) = -\frac{f_A\left(\vec{p}\right)}{\tau\left(\vec{p}\right)}$$

$$\sum_{p'} S\left(\vec{p},\vec{p}'\right) = \frac{1}{\tau\left(\vec{p}\right)}$$ "out-scattering rate"

isotropic scattering: RTA valid, $\tau_f = \tau = \tau_m$

RTA (iii)

$$\hat{C}f\left(\vec{p}\right) = -f_A\left(\vec{p}\right)\sum_{p'} S\left(\vec{p},\vec{p}'\right)\left(1 - \frac{f_0\left(\vec{p}\right)}{f_0\left(\vec{p}'\right)}\frac{f_A\left(\vec{p}'\right)}{f_A\left(\vec{p}\right)}\right)$$

ii) elastic scattering: $f_0\left(\vec{p}\right) = f_0\left(\vec{p}'\right)$

"it can be shown…"

$$\hat{C}f = -f_A\left(\vec{p}\right)\sum_{p'} S\left(\vec{p},\vec{p}'\right)\left(1 - \cos\alpha\right) = -\frac{f_A\left(\vec{p}\right)}{\tau_f}$$

$$\frac{1}{\tau_f} = \sum_{p'} S\left(\vec{p},\vec{p}'\right)\left(1 - \cos\alpha\right)$$ "momentum relaxation rate"

elastic scattering: RTA valid, $\tau_f = \tau_m$

RTA (iv)

The (microscopic) momentum relaxation time is valid:

1) Near equilibrium

2) For isotropic scattering or elastic scattering.

When valid, the characteristic time is the momentum relaxation time.

See Lundstrom, Chapter 3, pp. 139-141 for more discussion.

questions

1) Characteristic times

2) Relaxation Time Approximation and τ_f