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Lecture 16: Solving the BTE: Magnetic Fields

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the BTE

$$f\left(\vec{r},\vec{p},t\right)$$

$$\frac{\partial f}{\partial t}+\vec{\upsilon}\bullet\nabla_r f+\vec{F}_e\bullet\nabla_p f=-\frac{\left(f-f_0\right)}{\tau_f}$$

1) 1, 2, and 3D with arbitrary bandstructures

2) Temperature gradients and electric fields

3) Magnetic fields

outline

1) **General solution**

2) Current equation

3) Coupled current equations

4) Example

5) Summary

BTE

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = \left.\frac{df}{dt}\right|_{coll}$$

1) near-equilibrium: $f(\vec{p}) = f_S(\vec{p}) + f_A(\vec{p}) \quad \left|f_S(\vec{p})\right| >> \left|f_A(\vec{p})\right|$

2) RTA: $\left. df/dt \right|_{coll} \approx -f_A/\tau_f$

3) steady-state transport: $\partial f/\partial t = 0$

4) spatially uniform
(just to keep math simple): $\nabla_r f = 0 \quad \nabla_r F_n \to \vec{\mathcal{E}}$

5) electric + (small) magnetic fields $\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}$

BTE

$$\left(-q\vec{\mathcal{E}} - q\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f_0 = -f_A/\tau_f$$

Solve by superposition:

1) first, assume *B* = 0 and solve for f'_A

2) then add *B* and solve for f''_A

(This procedure works for small magnetic fields.)

but first.....consider only a B-field

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f = -f_A/\tau_f$$

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f_S = -f_A/\tau_f$$

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\frac{\partial f_S}{\partial E}\nabla_p E = -f_A/\tau_f$$

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\vec{\upsilon}\left(\frac{\partial f_S}{\partial E}\right) = -f_A/\tau_f$$

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\vec{\upsilon} = 0$$

so we need to be careful about replacing *f* by f_S on the LHS!

first step: $B = 0$

$$-q\vec{\mathcal{E}} \bullet \nabla_p f_S = -\frac{f_A'(\vec{p})}{\tau_f}$$

$$f_A'(\vec{p}) = q\tau_f \vec{\mathcal{E}} \bullet \nabla_p f_S$$

$$\nabla_p f_S = \frac{\partial f_S}{\partial E} \nabla_p E = \frac{\partial f_S}{\partial E} \vec{\upsilon}$$

$$f_A'(\vec{p}) = -q\tau_f \left(-\frac{\partial f_S}{\partial E}\right) \vec{\upsilon} \bullet \vec{\mathcal{E}}$$

now add B

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f = -f_A''/\tau_f$$

$$\left(-q\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p\left(f_S + f_A'\right) = -f_A''/\tau_f$$

$$f_A'' = q\tau_f\left(\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f_A'$$

$$\nabla_p f_A' = \nabla_p\left\{-q\tau_f\left(-\frac{\partial f_S}{\partial E}\right)\vec{\upsilon}\bullet\vec{\mathcal{E}}\right\}$$

now add *B* (ii)

$$\nabla_p f_A' = \nabla_p \left\{ -q\tau_f \left(-\frac{\partial f_S}{\partial E} \right) \vec{\upsilon} \bullet \vec{\mathcal{E}} \right\}$$

$$f_A'' = q\tau_f \left(\vec{\upsilon} \times \vec{B} \right) \bullet \nabla_p f_A'$$

Exercise: show that only 1 term will give a non-zero contribution:

$$\nabla_p f_A' = \left\{ -q\tau_f (E) \left(-\frac{\partial f_S}{\partial E} \right) \left[\nabla_p \left(\vec{\upsilon} \bullet \vec{\mathcal{E}} \right) \right] \right\}$$

now add B (iii)

$$f_A'' = q\tau_f \left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p f_A'$$

$$\nabla_p f_A' = \left\{ -q\tau_f \left(E\right) \left(-\frac{\partial f_S}{\partial E} \right) \left[\nabla_p \left(\vec{\upsilon} \bullet \vec{\mathcal{E}} \right) \right] \right\}$$

$$\nabla_p \left(\vec{\upsilon} \bullet \vec{\mathcal{E}} \right) = ?$$

$$\nabla_p \left(\vec{\upsilon} \bullet \vec{\mathcal{E}} \right) = \frac{1}{m^*} \vec{\mathcal{E}}$$

$$\nabla_p \left(\upsilon_x \mathcal{E}_x + \upsilon_y \mathcal{E}_y + \upsilon_z \mathcal{E}_z \right) = \frac{\partial}{\partial p_x} \left(\upsilon_x \mathcal{E}_x \right) \hat{x} + \frac{\partial}{\partial p_y} \left(\upsilon_y \mathcal{E}_y \right) \hat{y} + \frac{\partial}{\partial p_z} \left(\upsilon_z \mathcal{E}_z \right) \hat{z}$$

$$\nabla_p \left(\upsilon_x \mathcal{E}_x + \upsilon_y \mathcal{E}_y + \upsilon_z \mathcal{E}_z \right) = \left(\frac{1}{m^*} \mathcal{E}_x \right) \hat{x} + \left(\frac{1}{m^*} \mathcal{E}_y \right) \hat{y} + \left(\frac{1}{m^*} \mathcal{E}_z \right) \hat{z}$$

now add *B* (iv)

first step: *B* = 0

$$-q\vec{\mathcal{E}} \bullet \nabla_p f_S = -\frac{f_A'(\vec{p})}{\tau_f}$$

$$f_A'(\vec{p}) = q\tau_f \vec{\mathcal{E}} \bullet \nabla_p f_S$$

$$\nabla_p f_S = \frac{\partial f_S}{\partial E} \nabla_p E = \frac{\partial f_S}{\partial E} \vec{\upsilon}$$

$$\boldsymbol{f_A'(\vec{p}) = -q\tau_f \left(-\frac{\partial f_S}{\partial E}\right) \vec{\upsilon} \bullet \vec{\mathcal{E}}}$$

solution

$$f\left(\vec{k}\right) = f_S\left(k\right) + f_A'\left(\vec{k}\right) + f_A''\left(\vec{k}\right)$$

$$f_S\left(k\right) = \frac{1}{1 + e^{\left(E - F_n\right)/k_B T}}$$

$$f_A'\left(\vec{p}\right) = q\tau_f \vec{\mathcal{E}} \bullet \nabla_p f_S$$

$$f_A'' = \frac{-q^2\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\left(\vec{\upsilon} \bullet \vec{B} \times \vec{\mathcal{E}}\right)$$

outline

1) General solution

2) Current equation

3) Coupled current equations

4) Example

5) Summary

electric current

$$\vec{J}_n = \frac{1}{\Omega}\sum_k (-q)\vec{\upsilon}\left[f_A'\left(\vec{k}\right) + f_A''\left(\vec{k}\right)\right] = \vec{J}_n' + \vec{J}_n''$$

$$J_i = J_i' + J_i'' = \sigma_{ij}'\mathcal{E}_j + \sigma_{ij}''\mathcal{E}_j = \sigma_{ij}\mathcal{E}_j$$

electric current: first component

$$J_i = \frac{1}{\Omega}\sum_k (-q)\upsilon_i\, f_A'(\vec{k}) \qquad J_i' = \sigma_{ij}'\mathcal{E}_j \qquad f_A'(\vec{p}) = q\tau_f\,\mathcal{E}_j\,\partial_{p_j} f_S$$

$$\sigma_{ij}' = \frac{q^2}{\Omega}\sum_k \upsilon_i\upsilon_j\tau_f\left(-\frac{\partial f_S}{\partial E}\right) \qquad \langle\langle\tau_f\rangle\rangle = \frac{\langle E\tau_f\rangle}{\langle E\rangle}$$

$$\tau_f(E) = \tau_0\left(E/k_BT\right)^s \rightarrow$$

$$\sigma_{ij}' = nq\frac{q\langle\langle\tau_f\rangle\rangle}{m^*}\delta_{ij} = \sigma_0\delta_{ij} \qquad \langle\langle\tau_f\rangle\rangle = \tau_0\frac{\Gamma(s+5/2)}{\Gamma(5/2)}$$

electric current: second component

$$J_i'' = \frac{1}{\Omega}\sum_k (-q)\upsilon_i f_A''(\vec{k}) \qquad f_A'' = \frac{-q^2\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\left(\vec{\upsilon}\bullet\vec{B}\times\vec{\mathcal{E}}\right)$$

$$J_i'' = \sigma_{ij}''\left(\vec{B}\right)\mathcal{E}_j$$

need to write the cross product in indicial notation….

$$\left(\vec{A}\times\vec{B}\right)\bullet\hat{x}_i = \varepsilon_{ijk}A_jB_k$$ (recall Lecture 10)

second component (ii)

$$J_i'' = \frac{1}{\Omega}\sum_k (-q)\upsilon_i f_A''(\vec{k}) = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_i\left(\vec{\upsilon}\bullet\vec{B}\times\vec{\mathcal{E}}\right)$$

$$\left(\vec{\upsilon}\bullet\vec{B}\times\vec{\mathcal{E}}\right) = ?$$

$$\left(\vec{B}\times\vec{\mathcal{E}}\right)\bullet\hat{x}_m = \varepsilon_{mnj}B_n\mathcal{E}_j$$

$$\left(\vec{\upsilon}\bullet\vec{B}\times\vec{\mathcal{E}}\right) = \upsilon_m\left(\varepsilon_{mnj}B_n\mathcal{E}_j\right)$$

$$\left(\vec{A}\times\vec{B}\right)\bullet\hat{x}_i = \varepsilon_{ijk}A_jB_k$$

second component (iii)

$$J_i'' = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_i\left(\upsilon_m\varepsilon_{mnj}B_n\mathcal{E}_j\right) \equiv \sigma_{ij}''\left(\vec{B}\right)\mathcal{E}_j$$

$$\sigma_{ij}''\left(\vec{B}\right) = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_i\upsilon_m\varepsilon_{mnj}B_n$$

First consider the diagonal terms, *i* = *j*, then the off-diagonal terms, *i* ≠ *j*.

diagonal terms

$$\sigma''_{ij}\left(\vec{B}\right) = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_i\upsilon_m\varepsilon_{mnj}B_n$$

i = *j* = 1:

$$\sigma''_{11}\left(\vec{B}\right) = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_1\upsilon_m\varepsilon_{mn1}B_n$$

Recall from Lecture 15, that when we integrate over θ and φ, only the *m* = 1 term survives.

$$\sigma''_{11}\left(\vec{B}\right) = \frac{1}{\Omega}\sum_k \frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_1\upsilon_1\varepsilon_{1n1}B_n$$

$$\varepsilon_{1n1} = 0$$

$$\sigma''_{11}\left(\vec{B}\right) = \sigma''_{22}\left(\vec{B}\right) = \sigma''_{33}\left(\vec{B}\right) = 0$$

Diagonal terms are not affected by *B* (to first order in *B*).

off-diagonal terms

$$\sigma''_{ij}\left(\vec{B}\right)=\frac{1}{\Omega}\sum_{k}\frac{q^{3}\tau_{f}^{2}}{m^{*}}\left(-\frac{\partial f_{S}}{\partial E}\right)\upsilon_{i}\upsilon_{m}\varepsilon_{mnj}B_{n}$$

i = 1 *j* = 2:

$$\sigma''_{12}\left(\vec{B}\right)=\frac{1}{\Omega}\sum_{k}\frac{q^{3}\tau_{f}^{2}}{m^{*}}\left(-\frac{\partial f_{S}}{\partial E}\right)\upsilon_{1}\upsilon_{m}\varepsilon_{mn2}B_{n}$$

$i)\quad m=3, n=1, \varepsilon_{312}=+1\rightarrow \upsilon_{1}\upsilon_{3}\rightarrow 0$

$ii)\quad m=1, n=3, \varepsilon_{132}=-1\rightarrow \upsilon_{1}\upsilon_{1}$

$$\sigma''_{12}\left(\vec{B}\right)=-\frac{1}{\Omega}\sum_{k}\frac{q^{3}\tau_{f}^{2}}{m^{*}}\left(-\frac{\partial f_{S}}{\partial E}\right)\upsilon_{1}^{2}B_{3}$$

off-diagonal term: evaluation

$$\sigma''_{12}\left(\vec{B}\right) = -\frac{1}{\Omega}\sum_{k}\frac{q^3\tau_f^2}{m^*}\left(-\frac{\partial f_S}{\partial E}\right)\upsilon_1^2 B_3$$

$$\sigma''_{12}\left(\vec{B}\right) = -\sigma_0 \mu_H B_3$$

$$\mu_H = r_H \mu_n \qquad r_H = \frac{\left\langle\left\langle \tau_f^2 \right\rangle\right\rangle}{\left\langle\left\langle \tau_f \right\rangle\right\rangle^2}$$

$$r_H = \frac{\Gamma\left(2s+5/2\right)\Gamma\left(5/2\right)}{\Gamma\left(s+5/2\right)^2}$$

Recall:

$$\sigma_{11} = \frac{q^2}{k_B T}\frac{1}{\Omega}\sum_{k}\upsilon_1^2\tau_f\left(-\frac{\partial f_S}{\partial E}\right)$$

non-degenerate:

$$\sigma_{11} = \sigma_0 = nq\mu_n$$

$$\mu_n = \frac{q\left\langle\left\langle \tau_f \right\rangle\right\rangle}{m^*} \qquad \left\langle\left\langle \tau_f \right\rangle\right\rangle = \frac{\left\langle E\tau_f \right\rangle}{\left\langle E \right\rangle}$$

$$\left\langle\left\langle \tau_f \right\rangle\right\rangle = \tau_0 \frac{\Gamma\left(s+5/2\right)}{\Gamma\left(5/2\right)}$$

final result

$$J_i = \sigma_{ij}\left(\vec{B}\right)\mathcal{E}_j$$

$$\sigma_{ij}\left(\vec{B}\right) = \sigma_0 \delta_{ij} - \sigma_0 \mu_H \varepsilon_{ijk} B_k$$

$$\sigma_{ij}\left(\vec{B}\right) = \sigma_0 \begin{bmatrix} 1 & -\mu_H B_z & +\mu_H B_y \\ +\mu_H B_z & 1 & -\mu_H B_x \\ -\mu_H B_y & +\mu_H B_x & 1 \end{bmatrix}$$

$$J_i = \sigma_0 \mathcal{E}_i - \sigma_0 \mu_H \varepsilon_{ijk} B_k \mathcal{E}_j$$

$$\vec{J} = \sigma_0 \vec{\mathcal{E}} - \sigma_0 \mu_H \vec{\mathcal{E}} \times \vec{B}$$

(In Lecture 18, we will present a simple derivation of this result.)

outline

1) General solution

2) Current equation

3) Coupled current equations

4) Example

5) Summary

coupled current equations (B = 0)

From Lecture 15:

$$J_i = \sigma_0 \mathcal{E}_i + [sg]_0 \partial_i T$$

$$J_i^Q = T[sg]_0 \mathcal{E}_i - \kappa_0 \partial_i T$$

$$\mathcal{E}_i = \rho_0 J_i + S_0 \partial_i T$$

$$J_i^Q = \pi_0 J_i - \kappa_0^e \partial_i T$$

Transport tensors were diagonal for parabolic energy bands.

coupled current equations ($B \neq 0$)

$$J_i = \sigma_{ij}\left(\vec{B}\right)\mathcal{E}_j + [sg]_{ij}\left(\vec{B}\right)\partial_j T$$

$$J_i^Q = T[sg]_{ij}\left(\vec{B}\right)\mathcal{E}_j - \kappa_{ij}^0\left(\vec{B}\right)\partial_j T$$

$$\mathcal{E}_i = \rho_{ij}\left(\vec{B}\right)J_j + S_{ij}\left(\vec{B}\right)\partial_j T$$

$$J_i^Q = \pi_{ij}\left(\vec{B}\right)J_{i=j} - \kappa_{ij}^e\left(\vec{B}\right)\partial_j T$$

All transport tensors now depend on the B-field.

form of the tensors

$$\mathcal{E}_i = \rho_{ij}\left(\vec{B}\right)J_j + S_{ij}\left(\vec{B}\right)\partial_j T$$

$$J_i^Q = \pi_{ij}\left(\vec{B}\right)J_j - \kappa_{ij}^e\left(\vec{B}\right)\partial_j T$$

$$\rho_{ij}\left(\vec{B}\right) = \rho_0 + \rho_0\mu_H\varepsilon_{ijk}B_k + \ldots$$

$$S_{ij}\left(\vec{B}\right) = S_0 + S_1\varepsilon_{ijk}B_k + \ldots$$

$$\pi_{ij}\left(\vec{B}\right) = \pi_0 + \pi_1\varepsilon_{ijk}B_k + \ldots$$

$$\kappa_{ij}^e\left(\vec{B}\right) = \kappa_0 + \kappa_1\varepsilon_{ijk}B_k + \ldots$$

See Smith, Janek, and Adler, Chapter 9, for expressions for the B-field dependent transport tensors.

outline

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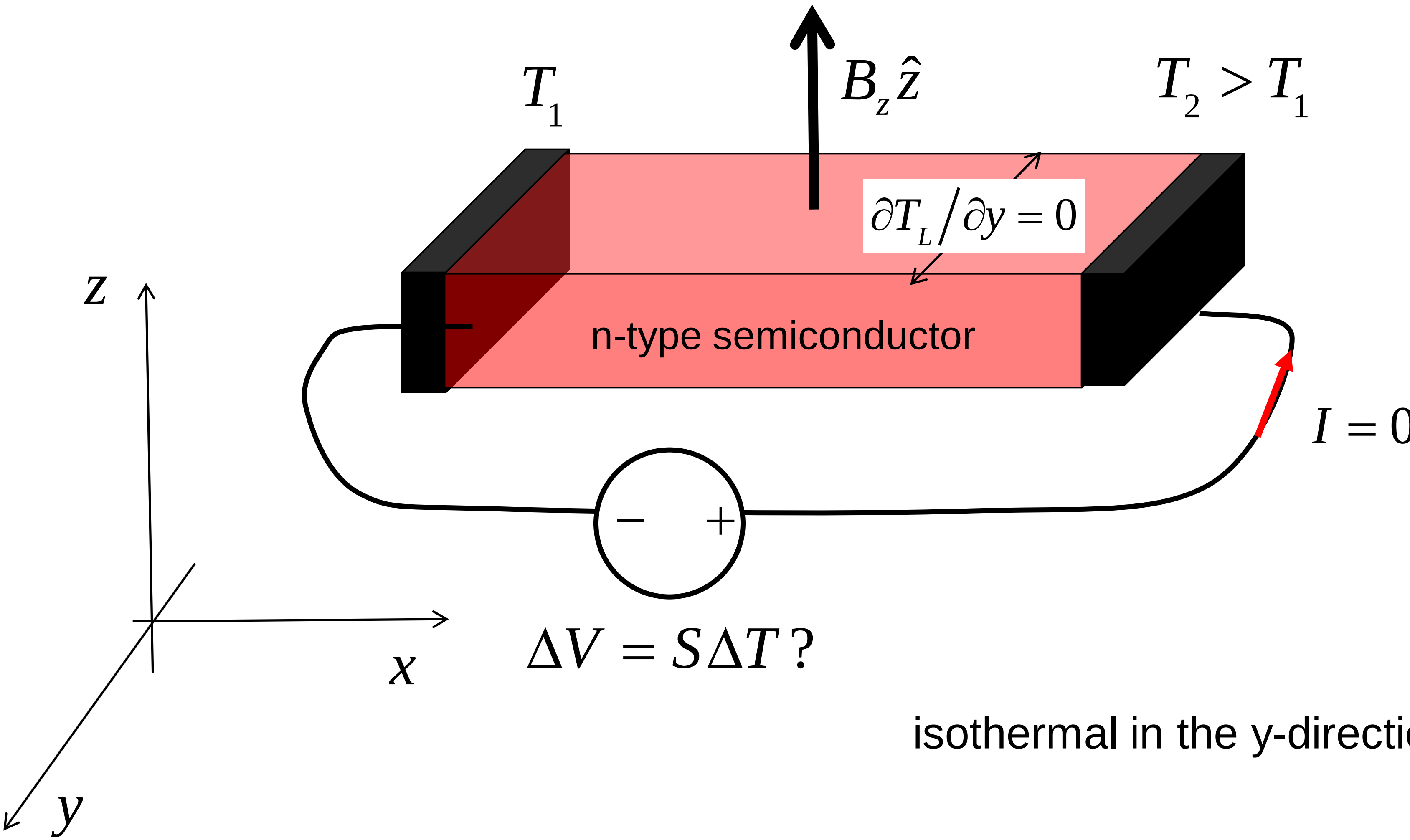
Seebeck and B-fields



$$\Delta V = S \Delta T\,?$$

isothermal in the y-direction

Magneto-Seebeck (isothermal in y)

$$\mathcal{E}_i = \rho_{ij}\left(\vec{B}\right)J_j + S_{ij}\left(\vec{B}\right)\partial_j T$$

$$J_i^Q = \pi_{ij}\left(\vec{B}\right)J_j - \kappa_{ij}^e\left(\vec{B}\right)\partial_j T$$

$$\mathcal{E}_x = S_0 \partial_x T$$

B-field has no effect (to first order)

$$\mathcal{E}_x = \rho_{ij}\left(\vec{B}\right)J_j + S_{xx}\left(\vec{B}\right)\partial_x T$$

$$\mathcal{E}_x = S_{xx}\left(\vec{B}\right)\partial_x T$$

$$S_{ij}\left(\vec{B}\right) = S_0 + S_1 \varepsilon_{ijk} B_k$$

$$S_{xx}\left(\vec{B}\right) = S_0 + S_1 \varepsilon_{xxz} B_z$$

$$S_{xx}\left(\vec{B}\right) = S_0$$

Magneto-Seebeck (adiabatic in y)

$$\mathcal{E}_i = \rho_{ij}\left(\vec{B}\right)J_j + S_{ij}\left(\vec{B}\right)\partial_j T$$

$$J_i^Q = \pi_{ij}\left(\vec{B}\right)J_j - \kappa_{ij}^e\left(\vec{B}\right)\partial_j T$$

$$J_y^Q = 0 = -\kappa_{yj}^e\left(\vec{B}\right)\partial_j T$$

$$\kappa_{ij}^e = \kappa_0^e\delta_{ij} + \kappa_1^e\varepsilon_{ijk}B_k$$

$$0 = -\kappa_0^e\left(\vec{B}\right)\partial_y T - \kappa_1^e\varepsilon_{yxz}\partial_x T\, B_z$$

$$\partial_y T = \frac{\kappa_1^e}{\kappa_0^e}B_z$$

$$\mathcal{E}_x = S_{xj}\left(\vec{B}\right)\partial_j T$$

$$S_{ij}\left(\vec{B}\right) = S_0\delta_{ij} + S_1\varepsilon_{ijk}B_k$$

$$\mathcal{E}_x = S_0\partial_x T + S_1 B_z\partial_y T$$

$$\mathcal{E}_x = \left(S_0 + \frac{S_1\kappa_1^e}{\kappa_0^e}B_z^2\right)\partial_x T$$

$$\mathcal{E}_x = S_{app}\partial_x T$$

comments

See Lundstrom, p. 179 for table listing other effects involving a transverse B-field.

(Also includes effects to second order in B-field)

exercise

Test your understanding of this lecture by working out the magneto-conductivity tensor for graphene.

questions

1) General solution

2) Current equation

3) Coupled current equations

4) Example

5) Summary