Fundamentals of Atomic Force Microscopy
Part 2: Dynamic AFM Methods

Week 1, Lecture 3
Point Mass Oscillators: Frequency Response

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From the last lecture

- Transient response of point mass models - overdamped/underdamped

\[ T = \frac{2\pi}{\omega_0 \sqrt{1 - \frac{1}{4Q^2}}} \]

- Point mass models of excited cantilevers

Force "directly" excites cantilever

Cantilever base oscillates

Effective external force \( F_{ext}(t) \)

\( q_{rel} = q - z_{base} \)

Photodiode observes \( q_{rel} \)
**Response of directly excited AFM levers**

\[ m \ddot{q} + c \dot{q} + k q = F_{\text{ext}}(t) \]

\[ \frac{\ddot{q}}{\omega_0^2} + \frac{1}{\omega_0 Q} \dot{q} + q = \frac{1}{k} F_{\text{ext}}(t) \]  

(1)

\[ F_{\text{ext}}(t) = F_0 \sin(\omega t) \]

with \( \omega_0 = \sqrt{\frac{k}{m}}, Q = \frac{m \omega_0}{c} \)

Measured motion = \( q(t) \)

\( \omega \) is excitation frequency not necessarily = \( \omega_0 \)

General solution is \( q(t) = q^p(t) + q^h(t) \) with \( q^p(t) = A \sin(\omega t - \phi) \) where (see Appendix)

\[
A = \frac{1}{\left(1 - r^2\right)^2 + \left(r/Q\right)^2}^{1/2}
\]

- the magnitude of transfer function (2)

\[ \phi(\omega) = \tan^{-1}\left(\frac{r}{Q(1 - r^2)}\right) \]

- phase lag of the transfer function (3)

where \( r = \frac{\omega}{\omega_0} \) and for under damped levers \( q^h(t) = \) transient solution (4)

\[
q^h(t) = e^{-\frac{\omega_0}{2Q}t} \left\{ q(0) \cos \left( \sqrt{1 - \frac{1}{4Q^2}} \omega_0 t \right) + \frac{\dot{q}(0) + \omega_0 \frac{q(0)}{2Q}}{\sqrt{1 - \frac{1}{4Q^2}}} \sin \left( \sqrt{1 - \frac{1}{4Q^2}} \omega_0 t \right) \right\}
\]
Response of directly excited AFM levers

\[
\frac{A}{F_0/k} = \left( \frac{1}{(1-r^2)^2 + (r/Q)^2} \right)^{1/2}
\]

\[
\phi(\omega) = \tan^{-1} \left( \frac{r}{Q(1-r^2)} \right)
\]

- Peak close to but slightly less than \(r=1\) (See Appendix)
- Phase lag = 90 deg always when \(r=1\) (See Appendix)
- Amplitude peak is asymmetric about \(r=1\)
- At \(r=1\), or when \(\omega = \omega_0\), we must have \(F_0 = kA/Q\) !! (Appendix)
Response of acoustically excited levers

$$m \ddot{q} + c \dot{q} + k(q - z_{\text{base}}) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad Q = \frac{m \omega_0}{c}$$

$$z_{\text{base}}(t) = Z_0 \sin(\omega t)$$

$$q_{\text{rel}}(t) = A \sin(\omega t - \phi_{\text{acoustic}})$$

$$\frac{A}{Z_0} = \frac{(1 + Q^2 r^2)}{\sqrt{(Q^2 - Q^2 r^2 - 1)^2 + Q^2 / r^2}}$$

$$\phi_{\text{acoustic}}(\omega) = \tan^{-1}\left(\frac{Q / r}{(Q^2 - Q^2 r^2 - 1)}\right)$$

where

$$r = \frac{\omega}{\omega_0}$$
Response of acoustically excited levers

\[ \frac{A}{Z_0} = \frac{(1 + Q^2 r^2)}{\sqrt{(Q^2 - Q^2 r^2 - 1)^2 + Q^2 / r^2}} \]

\[ \phi_{\text{acoustic}}(\omega) = \tan^{-1}\left(\frac{Q/r}{(Q^2 - Q^2 r^2) - 1}\right) \]

- Peak close to but slightly greater than \( r=1 \) (See Appendix)
- For large \( Q \), phase lag \( \sim 90 \) deg when \( r=1 \) (See Appendix)
- Amplitude peak is asymmetric about \( r=1 \)
- At \( r=1 \), or when \( \omega = \omega_0 \), we must have \( Z_0 \sim A/Q \) !! (Appendix)
How to tell if your system measures phase lead or phase lag?
Some key results for directly excited cantilevers

\[ \frac{A}{F_0/k} = \left( \frac{1}{(1-r^2)^2 + (r/Q)^2} \right)^{1/2} \]

\[ \phi(\omega) = \tan^{-1} \left( \frac{r}{Q(1-r^2)} \right) \]  \hspace{1cm} (8)

- When \( \omega = \omega_0 \), then setting \( r = 1 \) in (8) we get

\[ \frac{A}{F_0/k} = Q \text{ or } F_0 = \frac{kA}{Q} \quad \text{and} \quad \phi = \tan^{-1}(+\infty) = \pi/2 \] \hspace{1cm} (9)

- Which excitation frequency \( r = \omega/\omega_0 \) yields max amplitude?

Max amplitude is reached when \( d\left( \frac{A}{F_0/k} \right)/dr = 0 \) or when

\[ -\left( \frac{1}{2} \right)(1-r^2)r + 2r/Q^2 = 0 \]

or when

\[ r = \sqrt{1 - \frac{1}{2Q^2}} \] \hspace{1cm} (10)

which implies that max amplitude is reached when \( \omega \) is a little less than \( \omega_0 \)!

- When cantilever is driven at frequency corresponding to its max amplitude

\[ \frac{A}{F_0/k} = \frac{Q}{\sqrt{1 - \frac{4}{Q^2}}} \]

\[ \phi = \tan^{-1} \left( \sqrt{4Q^2 - 2} \right) \] \hspace{1cm} (11)

which implies that max amplitude corresponds to a phase lag a little less than \( \pi/2 \)!

- Fortunately if \( Q \geq 10 \), only a small error is incurred (\( \leq 3\% \))

by assuming that \( \phi = \pi/2 \) when excited at freq. corresponding to max. amplitude

For \( Q = 100 \) the error is even smaller \( \ll 0.3\% \), so the assumption is fine for operation in air or vacuum

but needs to be taken into account in liquids where \( Q < 10 \)
Some key results for acoustically excited levers

\[
\frac{A}{Z_0} = \sqrt{\frac{r^4 + r^2 / Q^2}{(1-r^2)^2 + r^2 / Q^2}}, \quad \phi_{\text{acoustic}}(\omega) = \tan^{-1}\left(\frac{Q/r}{Q^2 - Q^2r^2 - 1}\right)
\]

(12)

- When \(\omega = \omega_0\), then setting \(r = 1\) in (12) we get

\[
\frac{A}{Z_0} = Q\sqrt{1 + 1/Q^2} \quad \text{and} \quad \phi = \tan^{-1}(+\infty) = \pi / 2
\]

(13)

- Which excitation frequency \(r = \omega / \omega_0\) yields max amplitude?

Max amplitude is reached when \(d(\frac{A}{Z_0})/dr = 0\)

or when \(r = \sqrt{\frac{1}{2} + \frac{1}{2} \left(\sqrt{1 + \frac{2}{Q^2}}\right)}\)

which implies that max amplitude is reached when \(\omega\) is a little less than \(\omega_0\)!

- When cantilever is driven at frequency corresponding to its max amplitude

\[
\frac{A}{F_0 / k} = \sqrt{\frac{Q}{1 - \frac{4}{Q^2}}} \quad \phi = \tan^{-1}\left(\sqrt{4Q^2 - 2}\right)
\]

which implies that max amplitude corresponds to a phase lag a little less than \(\pi/2\)!

- Fortunately if \(Q \geq 10\), only a small error is incurred (~ \(\leq 3\%\)) by assuming that \(\phi = \pi/2\) when excited at freq. corresponding to max. amplitude

For \(Q = 100\) the error is even smaller \(\sim 0.3\%\), so the assumption is fine for operation in air or vacuum but needs to be taken into account in liquids where \(Q < 10\)
How to tell if your AFM system measures phase lead or phase lag?

1. Perform a tuning curve
2. Examine phase response

If phase increases across resonance peak then the system measures phase lag

If phase decreases across resonance peak then the system measures phase lead