Introduction to Bioelectricity

Week 6: Numerical methods of solving differential equations
Lecture 6.1: Discrete-time solutions to continuous-time problems

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Week 6: Numerical methods

- Lecture 6.1: Discrete-time solutions
Lecture 6.1: Discrete-time solutions

- Types of equations:
  - Equation
    \[ y = f(x) \]
  - Differential equation
  - 2\textsuperscript{nd} order differential equation
  - Ordinary differential equation (ODE)
  - Homogeneous linear ODE
Lecture 6.1: Discrete-time solutions

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Lecture 6.1: Discrete-time solutions

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    \[ y = f(x) \]
  - **Differential equation**
    \[ \frac{dy}{dx} = f(x) \]
  - **2\text{nd} order differential equation**
    \[ \frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, x\right) \]
  - **Ordinary differential equation (ODE)**

- **Homogeneous linear ODE**
Lecture 6.1: Discrete-time solutions

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    \[ \frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, x\right) \quad \text{vs} \quad \frac{d^2y}{dt^2} = f\left(\frac{dy}{dt}, \frac{dy}{dx}, x\right) \]
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  • Homogeneous linear ODE
    \[ y = f(x_1 + x_2) = f(x_1) + f(x_2) \]
Lecture 6.1: Discrete-time solutions

• Types of equations:
  • Equation
    \[ J_m = J_c + J_{K^+} + J_{Na^+} + J_L \]
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  • 2\textsuperscript{nd} order differential equation
    \[ \frac{1}{2\pi a(r_o + r_i)} \frac{\partial^2 V_m(z,t)}{\partial z^2} = \frac{C_m}{\partial t} \frac{\partial V_m}{\partial t} + G_{K^+} (V_m - V_{K^+}) + G_{Na^+} (V_m - V_{Na^+}) + G_L (V_m - V_L) \]
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  - Ordinary differential equation (ODE)
    \[ \frac{1}{v^2 2\pi a(r_0 + r_i)} \frac{d^2 V_m(z,t)}{dt^2} = C_m \frac{dV_m}{dt} + G_{K^+} (V_m - V_{K^+}) + G_{Na^+} (V_m - V_{Na^+}) + G_L (V_m - V_L) \]
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  • Homogeneous linear ODE
    \[ \frac{1}{v^2 2\pi a(r_o + r_i)} \frac{d^2 V_m(z,t)}{dt^2} = f(J_1 + J_2 + J_3 + ...) = f(J_1) + f(J_2) + f(J_3) + ... \]
Lecture 6.1: Discrete-time solutions

• Analytical solutions:

\[ \frac{dy}{dx} = f(x) = \cos(x) \]
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  \]
  \[
  \therefore y = \sin(x) + C
  \]

- **Or:**
  \[
  \lambda^2 \frac{\partial^2 v_m(z,t)}{\partial z^2} = v_m(z,t) + \tau \frac{\partial v_m(z,t)}{\partial t} - \lambda^2 r_o k_e(z,t)
  \]
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  \]

  \[
  \therefore v_m(z,t) = \frac{\lambda r_o Q_e}{\tau} \frac{1}{\sqrt{4\pi(t/\tau)}} e^{-\left(\frac{z}{\lambda}\right)^2/(4t/\tau)} e^{-t/\tau}
  \]
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- Discrete-time solutions:

![Graph showing discrete-time solutions with time in ms on the x-axis and voltage (V) on the y-axis. The graph shows a pattern with peaks at time 1 and a flattening out at times 2 and 3.]
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- Discrete-time solutions:
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• New variables:

- \( N \) = number of samples
- \( n \) = sample number from 0 through \( N-1 \)
- \( T \) = total interval sampled
- \( h \) = separation between samples = \( T/(N-1) \)
- \( V \) = dependent variable (in this example)
- \( t \) = independent variable (in this example)
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• Sampling rate:
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• **Sampling rate:**

• **Aliasing:**
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• Nyquist-Shannon sampling theorem:
  • Signals of interest are band-limited
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- Nyquist-Shannon sampling theorem:
  - Sampling frequency $f_{\text{sampling}} \geq 2f_s$
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- Nyquist-Shannon sampling theorem:
  - Noise is not band limited

![Diagram showing signal and noise bands with frequency axis from -f_s to f_s](image-url)
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- Nyquist-Shannon sampling theorem:
  - To avoid aliasing noise, we need to filter
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- Nyquist-Shannon sampling theorem:
  - New bandwidth is determined by filter order
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- Nyquist-Shannon sampling theorem:
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\[ f_{\text{sampling}} \geq 2f_c \]

\[ f_{\text{sampling}} = \frac{1}{h} = \frac{N - 1}{T} \geq 2f_c \approx 5 \text{kHz}^* \]
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• Nyquist-Shannon sampling theorem:

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• Why oversample?
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• Why oversample?
  – Tethered device power is cheap
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  – Tethered device power is cheap
  – Lower order anti-aliasing filter
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  – Lower and higher frequency quantization noise
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• Why not oversample?
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- Why oversample?
  - Tethered device power is cheap
  - Lower order anti-aliasing filter
  - Improved effective resolution (averaging)
  - Lower and higher frequency quantization noise

- Why not oversample?
  - Implantable device size dominated by battery