Introduction to Bioelectricity

Week 6: Numerical methods of solving differential equations
Lecture 6.2: Euler method

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Week 6: Numerical methods

• Lecture 6.2: Euler method
Lecture 6.2: Euler method

• Leonhard Euler (1707-1783)
  • Physicist:
    – Astronomy (parallax)
    – Mechanics (beam theory)
    – Optics (wave theory)
    – Music theory

• Mathematician
  – Notation
  – Analytical solutions of real-world problems
Lecture 6.2: Euler method

• Euler method of numerical solution:
  • Equation is of the type
  \[
  \frac{dy}{dt} = y' = f(y, t)
  \]
Lecture 6.2: Euler method

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  - Equation is of the type
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  - Problem is of the type:
    \[
    y_{n+1} = y_n + hf(y, t)
    \]
Lecture 6.2: Euler method

- Euler method of numerical solution:
  - Equation is of the type
    \[ \frac{dy}{dt} = y' = f(y, t) \]
  - Problem is of the type:
    \[ y_{n+1} = y_n + hf(y, t) \]
  - Euler assumes straight lines
    \[ \therefore y_{n+1} = y_n + hy'_n \]
Lecture 6.2: Euler method

• Euler method of numerical solution:

\[ y_{n+1} = y_n + h y'_n \]
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- Euler method of numerical solution:
  \[ y_{n+2} = y_{n+1} + hy'_{n+1} \]
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- Euler method of numerical solution:

\[ y_{n+3} = y_{n+2} + h y'_{n+2} \]
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• Euler method of numerical solution:

\[ y_{n+4} = y_{n+3} + hy'_{n+3} \]
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- Euler method of numerical solution:
  - Reconstruction
Lecture 6.2: Euler method

- Euler method of numerical solution:
  - Example:
  
  \[
  \frac{dy}{dt} = y + t^3 - 2
  \]
  
  Given
  
  \[
  y(0) = 0
  \]
  
  \[
  0 \leq t \leq 4
  \]
  
  \[
  N = 41
  \]
Lecture 6.2: Euler method

- Euler method of numerical solution:
  - Example:
    \[
    \frac{dy}{dt} = y + t^3 - 2
    \]
    Given
    \[
    y(0) = 0
    \]
    \[
    0 \leq t \leq 4
    \]
    \[
    N = 41
    \]
  - First:
    \[
    h = \frac{T}{N - 1} = \frac{4s}{40} = 0.1s
    \]
Lecture 6.2: Euler method

- **Euler method of numerical solution:**
  - Example:

\[
\frac{dy}{dt} = y + t^3 - 2
\]

\[y(0) = 0\]

\[0 \leq t \leq 4\]

\[N = 41\]

- **Second:**

\[y_{n+1} = y_n + hy'_n\]

\[\therefore y_1 = y_0 + hy'_0 = 0 + 0.1(y + t^3 - 2) = 0 + 0.1(0 + 0.1^3 - 2)\]

\[= -0.1999\]
Lecture 6.2: Euler method

- Euler method of numerical solution:
  - Example:

\[
\frac{dy}{dt} = y + t^3 - 2
\]
\[
y(0) = 0
\]
\[
0 \leq t \leq 4
\]
\[
N = 41
\]

- Third:

\[
y_{n+2} = y_{n+1} + hy'_{n+1}
\]

\[
\therefore y_2 = y_1 + hy'_1 = -0.1999 + 0.1(-0.1999 + 0.2^3 - 2)
\]

\[
= -0.41909
\]
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• Euler method observations
  • No calculus!

• Sources of error
  – Truncation error
    » Straight line assumption is wrong
    \[ y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + \cdots \]  
    Taylor expansion
    Truncation error
    » To lower error, decrease h
  – Round-off error
    » Every floating point calculation introduces error
    » To lower error, increase h
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- Taylor method
  - decrease truncation error
  - without increasing round-off error

\[ y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{3!} y'''_n + \cdots + \frac{h^\infty}{\infty!} y^\infty_n \]  

2nd order solution  \hspace{3cm} \text{Truncation error}

\[ y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{3!} y'''_n + \cdots + \frac{h^\infty}{\infty!} y^\infty_n \]  

3rd order solution  \hspace{3cm} \text{Truncation error}

- Add loads of calculus, sometimes impossible
Lecture 6.2: Euler method

• Taylor method of 2\textsuperscript{nd} order solution:
  • Example:

\[
\frac{dy}{dt} = y + t^3 - 2
\]

\[y(0) = 0\]

\[0 \leq t \leq 4\]

\[N = 41\]
Lecture 6.2: Euler method

• Taylor method of 2\textsuperscript{nd} order solution:

\[
\frac{dy}{dt} = y + t^3 - 2
\]

- Example:

Given:

\[
y(0) = 0
\]

\[
0 \leq t \leq 4
\]

\[
N = 41
\]

Calculated:

\[
h = \frac{T}{N - 1} = \frac{4s}{40} = 0.1s
\]

\[
\frac{dy}{dt} = y' = y + t^3 - 2
\]

\[
\therefore y'' = 2t^2 + 1
\]
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• Taylor method of 2\textsuperscript{nd} order solution:
  
  \[ \frac{dy}{dt} = y + t^3 - 2 \]

  \( y(0) = 0 \)

  \( 0 \leq t \leq 4 \)

  \( N = 41 \)

  • Example:
    
    \[
    h = \frac{T}{N - 1} = \frac{4s}{40} = 0.1s
    \]

    \[
    \frac{dy}{dt} = y' = y + t^3 - 2
    \]

    \[
    \therefore y'' = 2t^2 + 1
    \]

    Given
    Calculated

    \[
    y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n
    \]

    \[
    y_1 = y_0 + hy'_1 + \frac{h^2}{2} y''_1 = 0 + 0.1(0 + 0.1^3 - 2) + \frac{0.1^2}{2}(2(0.1)^2 + 1)
    \]
Lecture 6.2: Euler method

• Final note on numerical instabilities
  • Let's say
    \[ y = -x^2 \]
    \[ y' = -2x \]
    \[ y(0) = 1 \]
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    \[ y = -x^2 \]
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  • Using Euler as an example
    \[ y_{n+1} = y_n + hy' = y_n - 2hy_n = (1 - 2h)y_n \]
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  • If
    \[ h > \frac{1}{2} \]
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  - Lets say
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  - Using Euler as an example
    \[ y_{n+1} = y_n + hy' = y_n - 2hy_n = (1 - 2h)y_n \]
  - If
    \[ h > \frac{1}{2} \]
  - Then
    \[ y_{n+1} > y_n !!! \]