Introduction to Bioelectricity

Week 3: Models of biological conductors
Lecture 3.5: Time-dependent solutions

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Week 3: Models of biological conductors

• Lecture 3.5: Time-dependent solutions

\[ v_m(z, t) = \frac{r_o \lambda Q_e / \tau_m}{\sqrt{4\pi (t / \tau_m)}} e^{-z/\lambda_c} e^{-4t/\tau_m} e^{-t/\tau_m} u(t) \]
Lecture 3.5: Time-dependent solutions

- Two modes of solving the cable equation:

\[ \lambda^2 \frac{\partial^2 v_m(z,t)}{\partial z^2} = v_m(z,t) + \tau \frac{\partial v_m(z,t)}{\partial t} - \lambda^2 r_o k_e(z,t) \]  

- Time independent \( \therefore \frac{\partial v_m}{\partial t} = 0 \)

- Time dependent \( \therefore \frac{\partial v_m}{\partial t} \neq 0 \)
Lecture 3.5: Time-dependent solutions

• For the time independent solution, assume that our stimulus $k_e$, is equal to the impulse function:

$$k_e(z, t) = i_e(t) = Q_e \delta(t)$$

• Where $Q_e$ is the total amount of charge delivered

Area under the curve $= Q_e$

Time

-2 -1 0 1 2 3
Lecture 3.5: Time-dependent solutions

• In this case:
  \[ \int_{t=-\infty}^{\infty} \int_{z=-\infty}^{\infty} k_e(z, t) \, dz \, dt = Q_e \]

• Let us also define normalized variables:
  \[ \lambda' = \frac{z}{\lambda} \quad \& \quad \tau' = \frac{t}{\tau} \]

• Therefore:
  \[ \nu_m(z, t) = \nu_m(\lambda' \lambda, \tau' \tau) = \nu_m(\lambda', \tau') \]

• Similarly:
  \[ k_e(z, t) = k_e(\lambda' \lambda, \tau' \tau) = k_e(\lambda', \tau') = \frac{Q_e}{\lambda \tau} \delta(\lambda', \tau') \]
Lecture 3.5: Time-dependent solutions

- We can now rewrite the cable equation:

\[
\lambda^2 \frac{\partial^2 v_m(z, t)}{\partial z^2} = v_m(z, t) + \tau \frac{\partial v_m(z, t)}{\partial t} - \lambda^2 r_o k_e(z, t)
\]

- To:

\[
\frac{\partial^2 v_m(\lambda', \tau')}{\partial \lambda'^2} = v_m(\lambda', \tau') + \frac{\partial v_m(\lambda', \tau')}{\partial \tau'} - \frac{Q_e}{\lambda \tau} \delta(\lambda', \tau')^2
\]
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• Finally, let us define a new variable:

\[ w_m(\lambda', \tau') = v_m(\lambda', \tau') e^{\tau'} \]

• Noting that:

\[ \frac{\partial^2 w_m(\lambda', \tau')}{\partial \lambda'^2} = \frac{\partial^2 v_m(\lambda', \tau')}{\partial \lambda'^2} e^{\tau'} \] (3)

• And:

\[ \frac{\partial w_m(\lambda', \tau')}{\partial \tau'} = v_m(\lambda', \tau') e^{\tau'} + \frac{\partial v_m(\lambda', \tau')}{\partial \tau'} e^{\tau'} \] (4)
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- Substituting (3) and (4) into (2):

\[
\frac{1}{e^{r'}} \frac{\partial^2 w_m(\lambda', \tau')}{\partial \lambda'^2} = \frac{1}{e^{r'}} \frac{\partial w_m(\lambda', \tau')}{\partial \tau'} - \frac{\lambda r_o Q_e}{\tau} \delta(\lambda', \tau')^{(5)}
\]

- If we set \( Q_e \) to zero, this reduces to:

\[
\frac{\partial^2 w_m(\lambda', \tau')}{\partial \lambda'^2} = \frac{\partial w_m(\lambda', \tau')}{\partial \tau'}
\]

- Which is of the form of the homogeneous diffusion equation.
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• Solving by inspection:

\[ w_m(\lambda', \tau') = \frac{A}{\sqrt{4\pi\tau'}} e^{-\frac{\lambda'^2}{4\tau'}} \]

\[ \therefore v_m(\lambda', \tau') = \frac{A}{\sqrt{4\pi\tau'}} e^{-\frac{\lambda'^2}{4\tau'}} e^{-\tau'} \]

\[ \therefore v_m(z, t) = \frac{A}{\sqrt{4\pi\left(\frac{t}{\tau}\right)}} e^{\left(\frac{z}{\lambda}\right)^2 / \left(4\frac{t}{\tau}\right)} e^{-\frac{t}{\tau}} \quad (6) \]
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• To find A, rearrange (5) into:

\[ \frac{\partial^2 w_m(\lambda', \tau')}{\partial \lambda'^2} = \frac{\partial w_m(\lambda', \tau')}{\partial \tau'} - \frac{\lambda r_o Q_e}{\tau} \delta(\lambda', \tau') e^\tau \]

• Integrate w.r.t. \( \lambda \)

\[ \int_{-\infty}^{\infty} \frac{\partial^2 w_m(\lambda', \tau')}{\partial \lambda'^2} d\lambda = \int_{-\infty}^{\infty} \frac{\partial w_m(\lambda', \tau')}{\partial \tau'} d\lambda - \frac{\lambda r_o Q_e}{\tau} \int_{-\infty}^{\infty} \delta(\lambda', \tau') e^\tau d\lambda \]

\[ \therefore \int_{\lambda'=-\infty}^{\infty} \frac{\partial w_m(\lambda', \tau')}{\partial \tau'} = \frac{\lambda r_o Q_e}{\tau} \delta(\tau') \]
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• Rearranging the order of integration:

\[
\frac{d}{d\tau} \int_{-\infty}^{\infty} w_m(\lambda, \tau') d\lambda = \frac{\lambda r_o Q_e}{\tau} \delta(\tau')
\]

\[
\frac{d}{d\tau} \int_{-\infty}^{\infty} w_m(\lambda', \tau) d\lambda = \frac{d}{d\tau} \int_{-\infty}^{\infty} \frac{A}{\sqrt{4\pi \tau'}} e^{-\lambda'^2/4\tau'} d\lambda = \frac{d}{d\tau} Au(\tau') = A\delta(\tau')
\]

\[
\therefore A = \frac{\lambda r_o Q_e}{\tau} \quad (7)
\]
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- Substituting (7) into (6)

\[ v_m(z, t) = \frac{\lambda r_o Q_e}{\tau} \frac{\tau}{\sqrt{4\pi \left( \frac{t}{\tau} \right)}} e^{\left( \frac{z}{\lambda} \right)^2 / \left( 4\pi / \tau \right)} e^{-t / \tau} \]