SPRING 2016    ECE 659 EXAM I

Friday, Jan.29, 2016, FNY B124, 230-320PM

NAME : _______________________________________________________

CLOSED BOOK

1 page of notes provided

All five questions carry equal weight
1.1. Describe how you obtain the relation

\[ I = \frac{q}{h} M \left( \mu^+ - \mu^- \right) \quad (A) \]

starting from the general expression

\[ I = \frac{q}{h} \int_{-\infty}^{+\infty} dE \tilde{M}(E) \left( f^+(E) - f^-(E) \right) \quad (B) \]

and obtain an expression for M. Please state your assumptions clearly.

**Solution:**

Assume

\[ f^\pm(E) = \frac{1}{1 + \exp \left( \frac{E - \mu^\pm}{kT} \right)} \]

\[ \bar{\mu} = \frac{1}{2} \left( \mu^+ + \mu^- \right) \]

and \( \mu^+ - \mu^- \ll kT \).

\[ f^+(E) - f^-(E) \approx \left( \frac{\partial f}{\partial \mu} \right)_{\mu = \bar{\mu}} (\mu^+ - \mu^-) = \left( -\frac{\partial f}{\partial E} \right) (\mu^+ - \mu^-) \]

Substituting into (B) we obtain

\[ I = \frac{q}{h} \int_{-\infty}^{+\infty} dE \tilde{M}(E) \left( -\frac{\partial f}{\partial E} \right) (\mu^+ - \mu^-) \]

\[ = M \]

which leads to (A) with M defined as shown.
1.2. For a 3D conductor (area: A, Length: L) with an energy-momentum relation

\[ E^2 = E_g^2 + v_0^2 p^2 \]

(a) Find the functions \( M(E) \), \( D(E) \) for positive \( E \) (\( E > 0 \)).

(b) Show that the following relation is satisfied:

\[ D(E)v(E)p(E) = N(E)d \]

**Solution:**

(a)

\[
N(E) = \frac{4\pi}{3} AL \left( \frac{p}{h} \right)^3 = \frac{4\pi}{3h^3} AL \left( \frac{E^2 - E_g^2}{v_0^2} \right)^{3/2} = \frac{4\pi}{3h^3v_0^3} AL \left( E^2 - E_g^2 \right)^{3/2}
\]

\[
D(E) = \frac{dN(E)}{dE} = \frac{4\pi}{h^3v_0^3} AL \left( E^2 - E_g^2 \right)^{1/2} E
\]

\[
M(E) = \pi A \left( \frac{p}{h} \right)^2 = \frac{\pi A}{h^2} \left( \frac{E^2 - E_g^2}{v_0^2} \right)
\]

(b) From (a),

\[
\frac{N(E)}{D(E)} = \frac{E^2 - E_g^2}{3E}
\]

Also,

\[
v(E)p(E) = \frac{dE}{dp} p = \frac{2v_0^2 p^2}{2E} = \frac{E^2 - E_g^2}{E}
\]

Hence,

\[ D(E)v(E)p(E) = N(E)d \]

The relation is satisfied.
1.3. Consider an otherwise ballistic channel with M modes having a scatterer in the middle where only a fraction $T$ of all the electrons proceed along the original direction, while the rest $(1-T)$ get turned around.

(a) Determine the values of $\mu^+$ and $\mu^-$ on either side of the scatterer in terms of $\mu_1$, $\mu_2$ and $T$ and (b) explain why the resistance associated with the voltage drop across the scatterer is given by $R_{\text{scatterer}} = \frac{\hbar}{q^2M} \frac{1-T}{T}$ while the total resistance is given by $R_{\text{total}} = \frac{\hbar}{q^2MT}$.

Solution:

(a) Since $I^{\pm} = (qM / \hbar) \mu^{\pm}$ we can write

$\mu_2^+ = T \mu_1^+ + (1-T) \mu_2^-$

$\mu_1^- = (1-T) \mu_1^+ + T \mu_2^-$

Assume $\mu_1^+ = \mu_1$ and $\mu_2^- = \mu_2$:

$\mu_2^+ = \mu_2 + T(\mu_1 - \mu_2)$

$\mu_1^- = \mu_1 - T(\mu_1 - \mu_2)$

(b) $I = I^+ - I^- = (qM / \hbar)(\mu_2^+ - \mu_2^-) = (qM / \hbar)(\mu_1^+ - \mu_1^-)$

$= (qMT / \hbar)(\mu_1 - \mu_2)$  \hspace{1cm} \text{Same answer on Left or Right}$R_{\text{total}} \equiv \frac{\mu_1 - \mu_2}{qI} = \frac{\hbar}{q^2MT}$

(c) $R_{\text{scatterer}} = \frac{1}{2} \frac{(\mu_1^+ + \mu_1^+ - (\mu_2^+ + \mu^-_2)}{qI} = \frac{\hbar}{q^2MT} \frac{1-T}{T}$
1.4. (a) A three terminal conductor is described by

\[ I_m = \frac{1}{q} \sum_n G_{mn} (\mu_m - \mu_n) \]

where \( G = \frac{q^2}{h} \begin{bmatrix} 10 & 6 & 4 \\ 4 & 10 & 6 \\ 6 & 4 & 10 \end{bmatrix} \)

Is this consistent with the requirement of current conservation? Explain

**Solution:** Current conservation (or Kirchoff’s law) requires

\[ 0 = \sum_m I_m = \frac{1}{q} \sum_{m,n} G_{mn} (\mu_m - \mu_n) = \frac{1}{q} \sum_{m,n} (G_{mn} - G_{nm}) \mu_m \]

Since this must be true regardless of the values of \( \mu_m \)

\[ \sum_n (G_{mn} - G_{nm}) = 0 \rightarrow \sum_n G_{mn} = \sum_n G_{nm} \]

The given G-matrix satisfies this requirement and hence is consistent with current conservation.

(b) A two terminal conductor is described by

\[ I_m = \frac{1}{q} \sum_n G_{mn} (\mu_m - \mu_n) \]

where \( G = \frac{q^2}{h} \begin{bmatrix} 10 & 6 \\ 4 & 10 \end{bmatrix} \)

Is this consistent with current conservation?

**Solution:** This G-matrix does not satisfy the above requirement and hence is not consistent with current conservation.
1.5. Evaluate the left hand side of the steady-state Boltzmann equation

\[ \nu_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z} \]

where \( f_0 \) is the equilibrium Fermi function with \( E = \epsilon(p_z) + U(z) \).

\[ f_0(E) \equiv \frac{1}{1 + \exp\left( \frac{E - \mu_0}{kT} \right)} \]

*Note*: \( \nu_z \equiv \frac{d\epsilon}{dp_z}, \ F_z \equiv -\frac{dU}{dz} \)

**Solution:**

\[ \nu_z \frac{\partial f_0}{\partial z} + F_z \frac{\partial f_0}{\partial p_z} = \frac{\partial f_0}{\partial E} \left( \nu_z \frac{\partial E}{\partial z} + F_z \frac{\partial E}{\partial p_z} \right) = \frac{\partial f_0}{\partial E} \left( \nu_z \frac{dU}{dz} + F_z \frac{d\epsilon}{dp_z} \right) = 0 \]

since \( \nu_z = \frac{d\epsilon}{dp_z}, \ F_z = -\frac{dU}{dz} \)