Nanophotonic Modeling
Lecture 4.7: Introduction to Finite Element Method (FEM)

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Finite Element Method

• In general, can formulate FEM problems as:
  \[ Lu = b \]
  – \( L \) is the stiffness matrix, representing overlap between basis functions
  – \( b \) is the integral of given PDE with respect to basis
  – \( u \) is unknown
Finite Element BPM

• Value of FEA comes from:
  – *Spatial flexibility*: can define each element to vary in size quite substantially
  – *Speed*: properly chosen basis functions have compact support, leading to a sparse matrix

• Disadvantages:
  – Greater complexity to treat each element
  – Often requires expensive matrix inversion
Finite Elements

- Shapes: 1D, 2D, and 3D

- Shape functions:
  
  1D: \( u(x) = \alpha + \beta x + \gamma x^2 + \cdots \)
  
  2D/3D: \( u(x) = \sum_{k=0}^{d} [\alpha_k x^k + \beta_k y^k + \gamma_k z^k] \)
Finite Elements

• Lagrange functions:

\[ \lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - \xi_0} \]

\[ \lambda_1(x) = \frac{x - \xi_0}{\xi_1 - \xi_0} \]

Basis functions \( \varphi_j(x) \) combine the Lagrange functions with compact support.

Finite Element Method

• Can define error function as:
  \[ E = Lu - b \]

• In order to eliminate errors, set weighted residual \( w_i \) in test space \( v \) to zero:
  \[ \int_v w_i (Lu - b) = 0 \]

• Galerkin’s method is a specific example of this:
  \[ \int_v \psi (Lu - b) = 0 \]
  where \( u(x) \) are the polynomials we saw earlier