Nanophotonic Modeling Lecture 4.8: Galerkin Method for Finite Element Problems

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Static Equilibrium

- Newton's Law for a 1D wire :

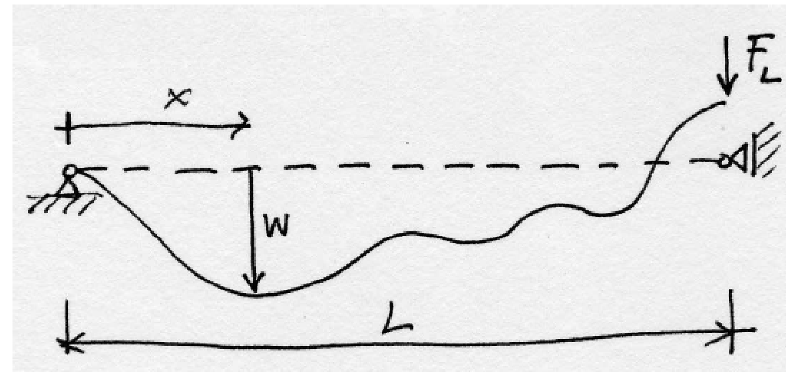
$$P\frac{\partial^2 w}{\partial x^2} + q = \mu\ddot{w}$$

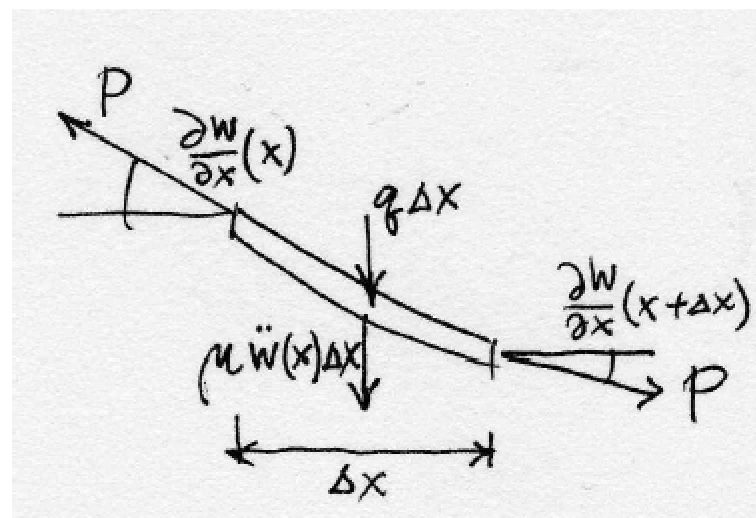
- In static equilibrium, forces balance exactly:

$$P\frac{\partial^2 w}{\partial x^2} + q = 0$$

- Define residual error in solution such that:

$$r_B = P\frac{\partial^2 w}{\partial x^2} + q$$





Galerkin Method

- In general, we want to apply Galerkin method with trial functions η_j:

$$\int_0^L dx\, \eta_j(x) r_B(x,t) = 0$$

- Analogy: stuff balloon into box, with each trial function a single 'finger'.
- Substituting: $\int_0^L dx\, \eta_j(x) \left[P \frac{\partial^2 w}{\partial x^2} + q \right] = 0$

Galerkin Method

- Integrating by parts:

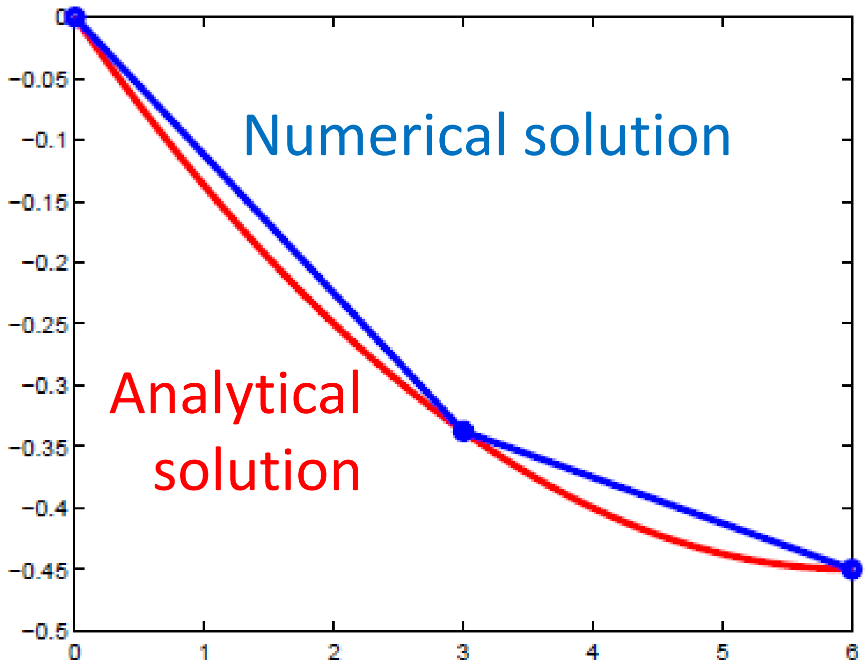
$$0 = \left[\eta_j P \frac{\partial w}{\partial x}\right]_0^L + \int_0^L dx \, \left[\eta_j q - \frac{\partial \eta_j}{\partial x} P \frac{\partial w}{\partial x}\right]$$

- Letting boundary terms vanish and substituting linear basis ($w = \sum_i N_i w_i$) yields:

$$0 = \int_0^L dx \, \eta_j q - \sum_{i=1}^{N} \left(\int_0^L dx \frac{\partial \eta_j}{\partial x} P \frac{\partial N_i}{\partial x} \right) w_i$$

$$0 = \boldsymbol{b} - \boldsymbol{K}\boldsymbol{w}$$

Static Equilibrium



Numerical solution matches analytical solution closely at key points

Dynamic Equilibrium

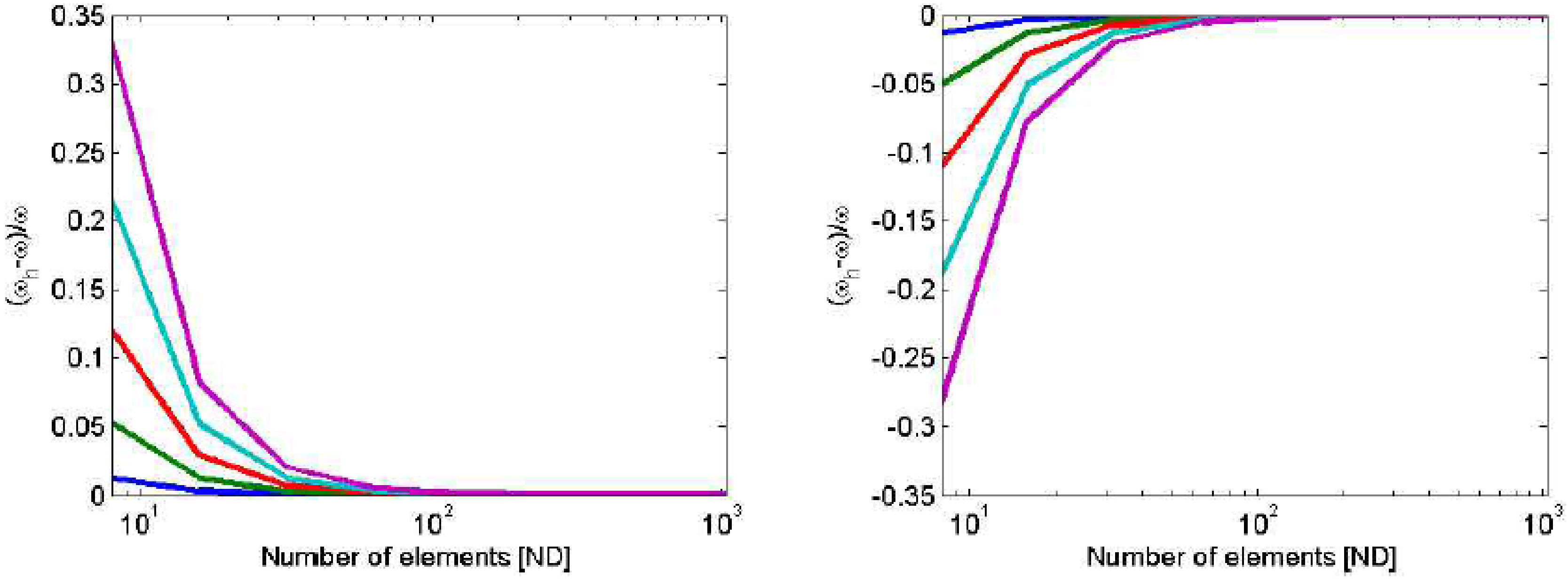
- Restoring time dependence will require tracking another second derivative term.
- In analogy with previous procedure, we can create another matrix, which yields:

$$0 = \boldsymbol{b} - \boldsymbol{K}\boldsymbol{w} - \boldsymbol{M}\ddot{\boldsymbol{w}}$$

- In absence of restoring force ($q = 0$), we have harmonic solutions ($\boldsymbol{w} = \boldsymbol{\phi} e^{i\omega t}$), so that:

$$\boldsymbol{K}\boldsymbol{\phi} - \omega^2 \boldsymbol{M}\boldsymbol{\phi} = 0$$

Dynamic Equilibrium



Convergence for first 5 frequencies as a function of the number of elements