Learning Objectives

By the Conclusion of this Lecture, You Should be Able to:

1. **Sketch** a simple Maxwell element, and utilize this construct to derive an expression for how the element would respond to an instantaneous stress.

2. **Define** what is meant by the storage and loss moduli of viscoelastic materials and differentiate between the storage modulus and loss modulus.

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A Maxwell Element is a Useful Model for Transient Response

The Maxwell element is composed of a spring that is an ideal, Hookean spring. This spring has a spring constant of $\hat{G}$. This spring is what is used to store the energy. That is, it provides the elastic response. The stress for the spring is as follows.

$$\sigma = \hat{G}\gamma$$

This spring is connected in series with a dashpot that has a viscosity of $\hat{\eta}$. The dashpot is what dissipates the energy and provides the viscous response for the element. The stress for the spring is the following.

$$\sigma = \hat{\eta}\dot{\gamma}$$

One can apply an instantaneous strain at zero time ($\gamma_0$) and hold this strain for a long period of time.

At first, the shock-absorbing dashpot will not want to move. However, the spring will take upon the initial deformation.

As time progresses, the spring will exert a force on the dashpot, and the system will eventually dissipate the energy through the viscous liquid.
The Response Follows an Exponential Decay

Mathematically, we can express the previous scenario as follows.

\[ \gamma_0 = \gamma_s + \gamma_d \]

Because the strain is held constant for times beyond the initial pulse, we know that:

\[ \frac{d\gamma}{dt} = 0 = \dot{\gamma}_s + \dot{\gamma}_d = \frac{d}{dt} \frac{\sigma(t)}{\hat{G}} + \frac{\sigma(t)}{\hat{\eta}} \]

We can then defining the relaxation time as follows. Physically, this is the time required to return to equilibrium after a disturbance.

\[ \tau \equiv \frac{\hat{\eta}}{\hat{G}} \]

This allows the above equation to be written as the following.

\[ \frac{d\gamma}{dt} = \frac{d}{dt} \frac{\sigma(t)}{\hat{G}} + \frac{\sigma(t)}{\hat{\eta}} = \dot{\sigma} + \frac{1}{\tau} \sigma \]

Here, \( \dot{\sigma} \) represents the time derivative of the stress variable. Thus, this equation is a linear, first-order ordinary differential equation.
The Response Follows an Exponential Decay

The solution to this equation is that of an exponential decay.

\[ \sigma(t) = \sigma_0 \exp \left[ -\frac{t}{\tau} \right] \]

Here, the pre-factor term is the initial stress response to the input strain.

In a similar manner, we can state that the stress relaxation modulus is as follows.

\[ G(t) = \frac{\sigma(t)}{\gamma_0} = \hat{G} \exp \left[ -\frac{t}{\tau} \right] \]

From a physical perspective, this means that the material will support the initial strain for a finite period of time, and then the viscous response will occur to cause the energy to dissipate.

While this is a useful model by which to think about how stress and strain work in transient systems, there is a more oft-employed means by which to evaluate this properties. This is through a sinusoidal transient function. This has the form of:

\[ \gamma(t) = \gamma_0 \sin \omega t \]

It can be shown that this leads to a differential equation of the following type.

\[ \dot{\sigma} + \frac{1}{\tau} \sigma = \hat{G} \gamma_0 \omega \cos \omega t \]
Solving this differential equation returns two coefficients.

These are the **storage modulus**:

\[
G' = \hat{G} \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2}
\]

And the **loss modulus**:

\[
G'' = \hat{G} \frac{\omega \tau}{1 + \omega^2 \tau^2}
\]

If both of these terms are non-trivial (i.e., one does not dominate), the material is said to be viscoelastic.

The material is more solid-like if the storage modulus is much larger than the loss modulus, and the material is more liquid-like when the loss modulus is larger than the storage modulus.

Often, one will discuss the cross-over point where the storage and loss modulus are equal.

**Next Time: Intrinsically-Stretchable Organic Field-Effect Transistors (OFETs)**