1) Working out Fermi-Dirac integrals just takes some practice. For practice, work out the integral

$$I_1 = \int_{-\infty}^{\infty} M(E) f_0(E) dE$$

where

$$f_0(E) = \frac{1}{1 + e^{(E-E_F)/k_B T}}$$

and

$$M(E) = \frac{2m^* (E-E_C)}{\pi \hbar} H(E-E_C)$$

where

$$H(E-E_C)$$ is the unit step function.

**Solution:**

$$I_1 = \int_{E_C}^{\infty} W \frac{\sqrt{2m^* (E-E_C)}}{\pi \hbar} \frac{1}{1 + e^{(E-E_F)/k_B T}} dE$$

Note that the unit step function in $M(E)$ makes the lower limit of the integral $E_C$.

$$I_1 = W \sqrt{2m^* \frac{(E-E_C)}{\hbar \pi}} \int_{E_C}^{\infty} \frac{(E-E_C)^{1/2}}{1 + e^{(E-E_F)/k_B T}} dE$$

Now make the change in variables, $\eta = (E-E_C)/k_B T$ and $\eta_F = (E_F-E_C)/k_B T$ to find:

$$I_1 = W \sqrt{2m^* \frac{k_B T \eta}{\pi \hbar}} \int_{0}^{\infty} (k_B T \eta)^{1/2} d\eta$$
ECE 656 Homework 1: Week 1 (continued)

\[ I_1 = W \sqrt{2m^* \pi \hbar} \left( k_B T \right)^{3/2} \int_0^\infty \frac{\eta^{1/2}}{1 + e^{\eta - \eta_F}} d\eta \]

Now we can recognize the FD integral of order \( \frac{3}{2} \):

\[ \int_0^\infty \frac{\eta^{1/2}}{1 + e^{\eta - \eta_F}} d\eta = \frac{\sqrt{\pi}}{2} F_{1/2}(\eta_F) \]

so the result becomes

\[ I_1 = W \sqrt{\frac{m^*/2\pi}{\hbar}} \left( k_B T \right)^{3/2} F_{1/2}(\eta_F) \]

which is the final answer.

2) For more practice, work out the integral in 1) assuming non-degenerate carrier statistics.

Solution:

We could approximate the Fermi function as

\[ f_0(E) = \frac{1}{1 + e^{(E - E_F)/k_B T}} \approx \exp\left(\frac{E - E_F}{k_B T}\right) \]

and then work out the integral

\[ I_1 = \int_{E_C}^{\infty} W \frac{2m^*(E - E_C)}{\pi \hbar} \frac{1}{1 + e^{(E - E_F)/k_B T}} dE \approx \int_{E_C}^{\infty} W \frac{2m^*(E - E_C)}{\pi \hbar} \exp\left(\frac{E - E_F}{k_B T}\right) dE \]

but it is easier to recognize that “non-degenerate” means \( E_F \ll E \) or \( \eta_F \ll 0 \) and that

\[ F_{1/2}(\eta_F) \approx \exp(\eta_F) \] for \( \eta_F \ll 0 \)

so we can use the result of prob. 1) and write the answer as

\[ I_1 = W \sqrt{\frac{m^*/2\pi}{\hbar}} \left( k_B T \right)^{3/2} \exp(\eta_F) \]
**ECE 656 Homework 1: Week 1 (continued)**

3) For still more practice, work out this integral:

\[ I_2 = \int_{E_c}^{\infty} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE , \]

where \( M(E) \) is as given in problem 1).

**Solution:**

From the form of the Fermi function, we see that

\[
\left( -\frac{\partial f_0}{\partial E} \right) = \left( +\frac{\partial f_0}{\partial E} \right)
\]

so the integral becomes

\[ I_2 = \int_{E_c}^{\infty} M(E) \left( +\frac{\partial f_0}{\partial E} \right) dE . \]

Since we are integrating with respect to energy, not Fermi energy, we can move the derivative outside of the integral to write

\[ I_2 = \frac{\partial}{\partial E_F} \int_{E_c}^{\infty} M(E) f_0(E) dE = \frac{1}{k_b T} \frac{\partial}{\partial (E_F/k_b T)} \int_{E_c}^{\infty} M(E) f_0(E) dE = \frac{1}{k_b T} \frac{\partial}{\partial \eta_F} \int_{E_c}^{\infty} M(E) f_0(E) dE \]

The integral can be recognized as the one we worked out in prob. 1), so

\[ I_2 = \frac{1}{k_b T} \frac{\partial}{\partial \eta_F} \left( \frac{1}{k_b T} \frac{\partial}{\partial \eta_F} \right) \left[ W \sqrt{m^* / 2\pi} \left( k_b T \right)^{3/2} F_{1/2} (\eta_F) \right] = W \sqrt{m^* k_b T / 2\pi} \frac{\partial}{\partial \eta_F} F_{1/2} (\eta_F) \]

Finally, using the differentiation property of FD integrals, we find

\[ I_2 = W \sqrt{m^* k_b T / 2\pi} F_{-1/2} (\eta_F) \]

The trick of replacing \(-\partial f_0 / \partial E\) with \(+\partial f_0 / \partial E_F\) and then moving the derivative outside of the integral is very useful in evaluating FD integrals.
ECE 656 Homework 1: Week 1 (continued)

4) It is important to understand when Fermi-Dirac statistics must be used and when non-degenerate (Maxwell-Boltzmann) statistics are good enough. The electron density in 1D is

\[ n_L = N_{1D} F_{-1/2}(\eta_F) \text{cm}^{-1}, \]

where \( N_{1D} \) is the 1D effective density of states and \( \eta_F = (E_F - E_C)/k_B T \). In 3D,

\[ n = N_{3D} F_{1/2}(\eta_F) \text{cm}^{-3}. \]

For Maxwell Boltzmann statistics

\[ n_{MB}^L = N_{1D} \exp(\eta_F) \text{cm}^{-1} \]
\[ n_{MB}^3 = N_{3D} \exp(\eta_F) \text{cm}^{-3}. \]

Compute the ratios, \( n_L/n_{MB}^L \) and \( n/n_{MB}^3 \) for each of the following cases:

a) \( \eta_F = -10 \)
b) \( \eta_F = -3 \)
c) \( \eta_F = 0 \)
d) \( \eta_F = 3 \)
e) \( \eta_F = 10 \)

Note that there is a Fermi-Dirac integral calculator available on nanoHUB.org. An iPhone app is also available.

Solution:

The iPhone app is called: “FD Integral”
The nanoHUB.org app is at: nanohub.org/resources/11396

a) \( \eta_F = -10 \):
\[ n_L/n_{MB}^L = F_{-1/2}(\eta_F) \exp(\eta_F) = F_{-1/2}(-10) / \exp(-10) = 4.54 \times 10^{-5} / 4.54 \times 10^{-5} = 1 \]
\[ n_L/n_{MB}^L = F_{-1/2}(-10) / \exp(-10) = 1 \]
\[ n/n_{MB}^3 = F_{1/2}(\eta_F) \exp(\eta_F) = F_{1/2}(-10) / \exp(-10) = 4.54 \times 10^{-5} / 4.54 \times 10^{-5} = 1 \]
\[ n/n_{MB}^3 = F_{1/2}(-10) / \exp(10) = 1 \]
ECE 656 Homework 1: Week 1 (continued)

b) \( \eta_F = -3 \):

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(-3) / \exp(-3) = \frac{4.81 \times 10^{-2}}{4.98 \times 10^{-2}} = 0.97
\]

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(-3) / \exp(-3) = 0.97
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(0) / \exp(0) = \frac{4.89 \times 10^{-2}}{4.98 \times 10^{-2}} = 0.98
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(0) / \exp(-3) = 0.98
\]

c) \( \eta_F = 0 \):

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(0) / \exp(0) = \frac{6.05 \times 10^{-1}}{1} = 0.61
\]

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(0) / \exp(-3) = 0.61
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(0) / \exp(0) = \frac{7.65 \times 10^{-1}}{1} = 0.77
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(0) / \exp(-3) = 0.77
\]

d) \( \eta_F = +3 \):

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(3) / \exp(3) = \frac{1.85 \times 10^0}{2.01 \times 10^1} = 0.092
\]

\[
\frac{n_L}{n_L^{MB}} = \mathcal{F}_{-1/2}(3) / \exp(3) = 0.092
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(3) / \exp(3) = \frac{4.49 \times 10^0}{2.01 \times 10^1} = 0.22
\]

\[
\frac{n}{n^{MB}} = \mathcal{F}_{+1/2}(3) / \exp(-3) = 0.22
\]
ECE 656 Homework 1: Week 1 (continued)

e) \eta_F = +3:

\[
\frac{n_L}{n_L^{MB}} = F_{-1/2} (\eta_F) / \exp(\eta_F) = F_{-1/2} (10) / \exp(10) = 3.55 \times 10^6 / 2.20 \times 10^4 = 1.610 \times 10^{-4}
\]

\[
\frac{n_L}{n_L^{MB}} = F_{-1/2} (10) / \exp(10) = 1.61 \times 10^{-4}
\]

\[
\frac{n}{n^{MB}} = F_{+1/2} (\eta_F) / \exp(\eta_F) = F_{+1/2} (10) / \exp(10) = 2.41 \times 10^7 / 2.20 \times 10^4 = 1.10 \times 10^{-3}
\]

\[
\frac{n}{n^{MB}} = F_{+1/2} (10) / \exp(10) = 1.10 \times 10^{-3}
\]

5) Consider GaAs at room temperature doped such that \( n = 10^{19} \) cm\(^{-3}\). The electron density is related to the position of the Fermi level according to

\[
n = N_C F_{1/2} (\eta_F) \text{ cm}^{-3}
\]

where

\[
N_C = 4.21 \times 10^{17} \text{ cm}^{-3}.
\]

Determine the position of the Fermi level relative to the bottom of the conduction band, \( E_C \).

a) assuming Maxwell-Boltzmann carrier statistics

b) NOT assuming Maxwell-Boltzmann carrier statistics

Solution:

a) \( n = N_C F_{1/2} (\eta_F) \rightarrow n = N_C \exp(\eta_F) \text{ cm}^{-3} \)

\[
\eta_F = \frac{E_F - E_C}{k_B T} = \ln \left( \frac{n}{N_V} \right) = \ln \left( \frac{10^{19}}{4.21 \times 10^{17}} \right) = 3.17
\]

\[
E_F = E_C + 3.17 \times 0.026 \text{ eV} = E_C + 0.082 \text{ eV}
\]

\[
E_F = E_C + 0.082 \text{ eV}
\]
ECE 656 Homework 1: Week 1 (continued)

b) \( n = N_C \mathcal{F}_{1/2}(\eta_F) \text{ cm}^{-3} \)

\[
\eta_F = \mathcal{F}_{1/2}^{-1}\left(n/N_C\right) = \mathcal{F}_{1/2}^{-1}\left(10^{19}/4.21 \times 10^{17}\right) = \mathcal{F}_{1/2}^{-1}(23.75) = 9.91
\]

\[
E_F = E_C + 9.91 \times 0.026 \text{ eV} = E_C + 0.26 \text{ eV}
\]

\[
E_F = E_C + 0.26 \text{ eV}
\]