Modern Physics

Unit 4: Heisenberg's Uncertainty Principle

Lecture 4.4: The Fourier Integral

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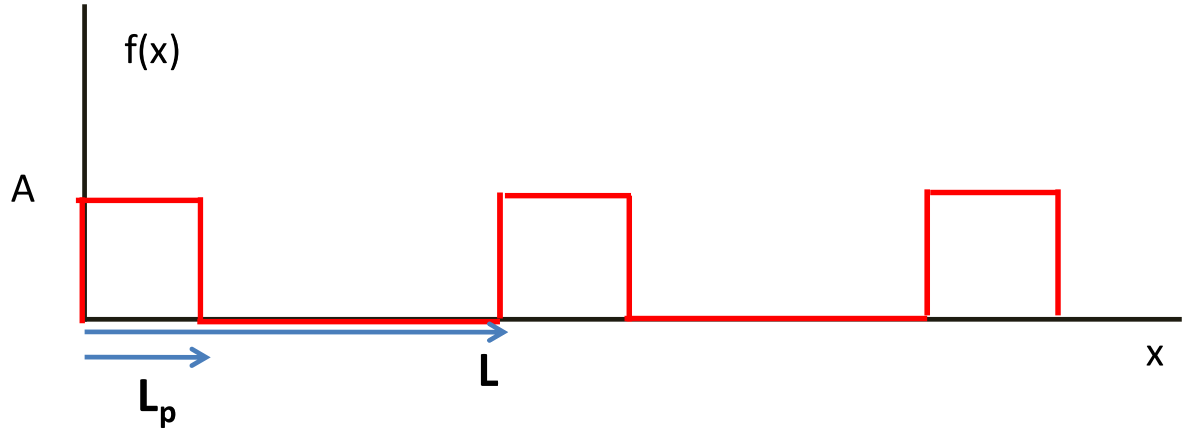
Fourier Series



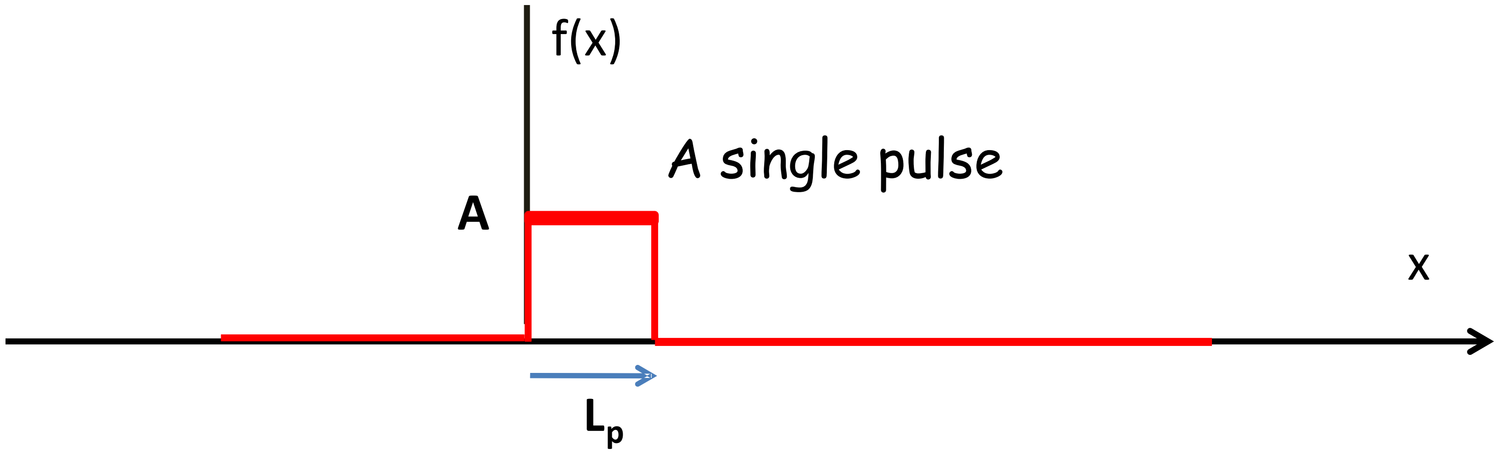
What happens when $L \gg L_p$?

$$f(x) = a_o + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi n x}{L} \right) + b_n \sin\left(\frac{2\pi n x}{L} \right) \right]$$

Progressively more terms are required in the Fourier Series before the waveform is adequately approximated.

In the limit, when the waveform consists of only ONE pulse (what's the periodicity?), a <u>continuous</u> distribution of sines and cosines are required.

The Fourier Transform



In this case, f(x) is no longer periodic. To reconstruct f(x), we now require a continuous distribution of sines and cosines (k is no longer discrete!). The sums become integrals. Instead of the a_n and b_n, f(x) is now specified by <u>some</u> <u>function</u> g(k) according to

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(k) e^{ikx} dk$$

Once f(x) is known, the function g(k) can be calculated from

$$g(k') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ik'x} dx$$

Note: This is now called a Fourier Transform, not a Fourier Series

Points to Ponder

$$g(k') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ik'x} dx$$

1. Remember, you know f(x) AND you must specify a value for k'
2. In general, $g(k')$ will be a complex number
3. The Fourier Transform is really an "ideal" model because
   - it requires a function f(x) that goes from $-\infty$ to $+\infty$
   - it requires an infinite number of k'

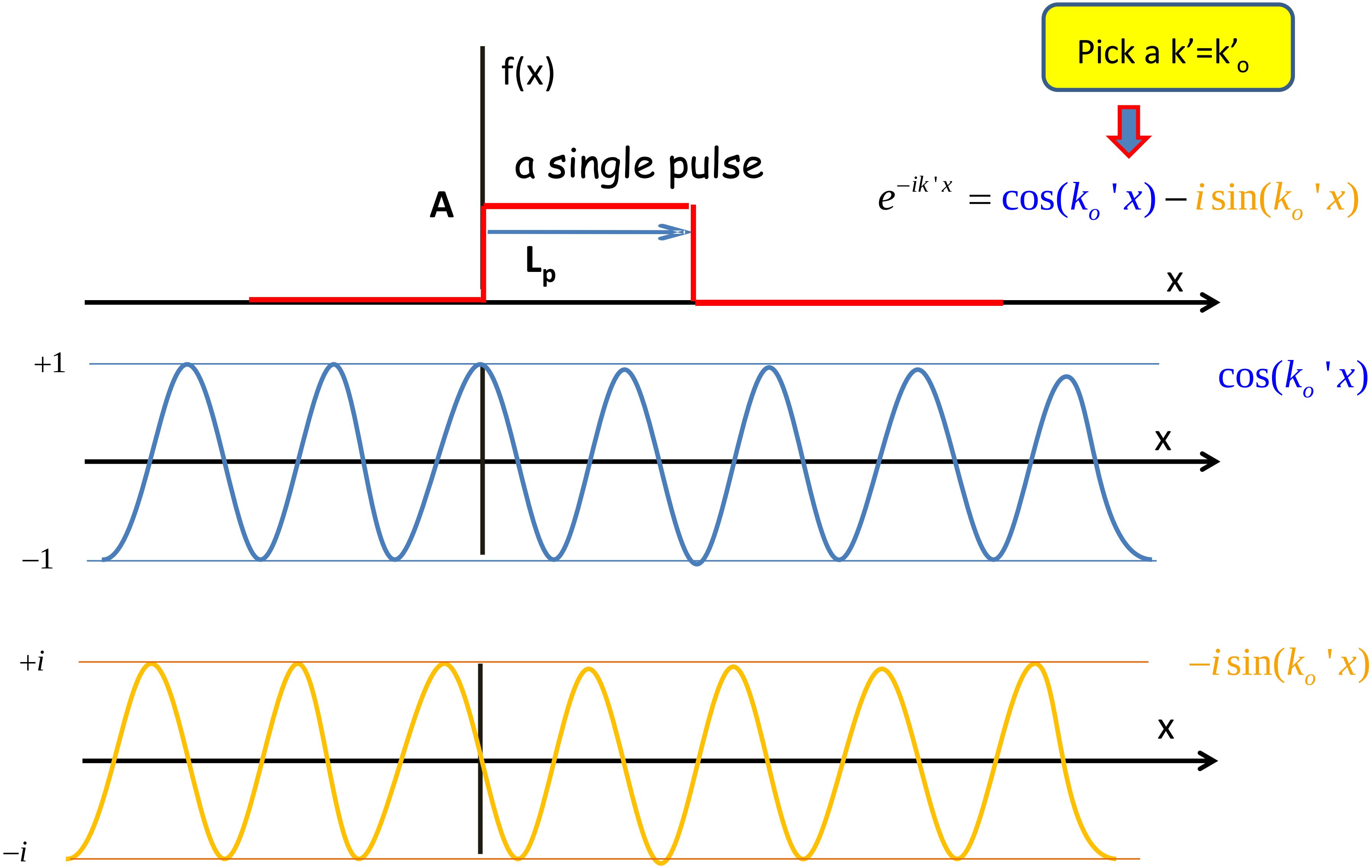
   like a double infinity

4. For every k' you specify, you need to perform the integral to get the value of $g(k')$ – so you need to perform an infinite number of integrals

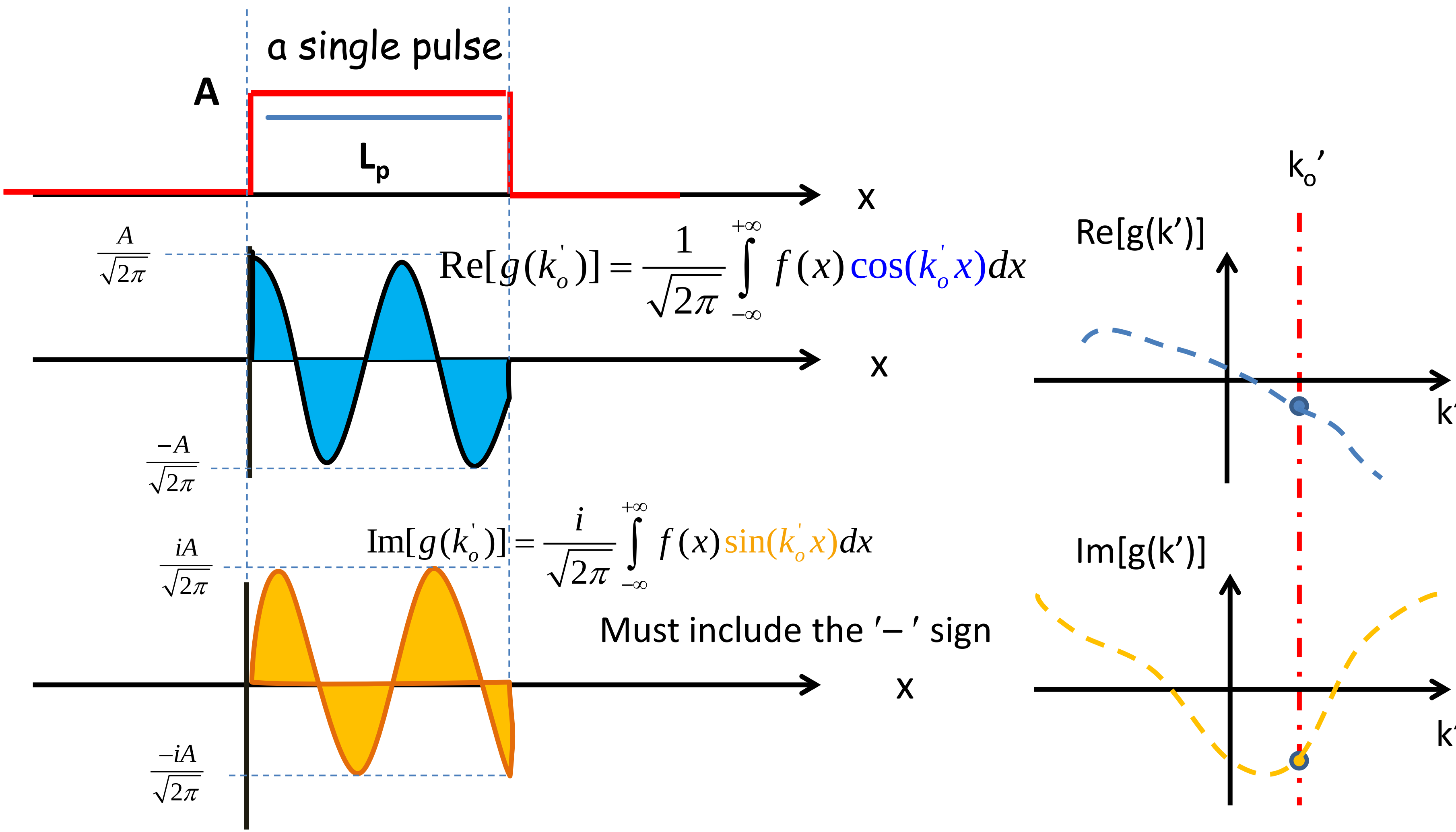
5. Sometimes, the integration works out so you have an **analytical form** for $g(k')$ – this is somewhat of a special case

6. In a Fourier Series, you can understand one term, but in a Fourier transform, it's very difficult to comprehend a single term because there are an infinite number of them, they span both positive and negative values of k' AND $g(k')$ for a specific k' can be a complex number

What the Fourier Transform does



Integrating Graphically



Working out the math

$$g(k') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ik'x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{L_p} A e^{-ik'x} dx$$

$$= \frac{A}{\sqrt{2\pi}} \frac{e^{-ik'x}}{-ik'}\Bigg|_0^{L_p} = \frac{A}{\sqrt{2\pi}} \left[\frac{e^{-ik'L_p} - 1}{-ik'} \right]$$

$$= \frac{A}{\sqrt{2\pi}} e^{-ik'L_p/2} \left[\frac{e^{-ik'L_p/2} - e^{ik'L_p/2}}{-ik'L_p/2} \right] \frac{L_p}{2} = \frac{A}{\sqrt{2\pi}} e^{-ik'L_p/2} \left[\frac{e^{ik'L_p/2} - e^{-ik'L_p/2}}{ik'L_p/2} \right] \frac{L_p}{2}$$

$$= \frac{AL_p}{\sqrt{2\pi}} e^{-ik'L_p/2} \left[\frac{e^{ik'L_p/2} - e^{-ik'L_p/2}}{2i} \right] \frac{1}{k'L_p/2}$$

$$= \frac{AL_p}{\sqrt{2\pi}} e^{-ik'L_p/2} \left[\frac{\sin\left(k'L_p/2\right)}{\left(k'L_p/2\right)} \right]$$

Note that now,
there is NO
restriction on k'

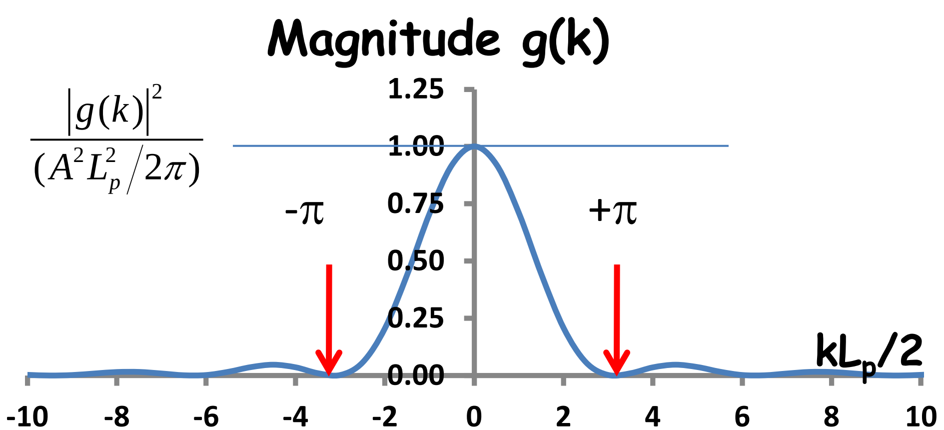
$$g(k) = \frac{AL_p}{\sqrt{2\pi}} e^{-ikL_p/2} \left[\frac{\sin\left(kL_p/2 \right)}{\left(kL_p/2 \right)} \right]$$

$$g(k) = \frac{AL_p}{\sqrt{2\pi}} \left[\frac{\sin\left(kL_p/2 \right)}{\left(kL_p/2 \right)} \right] \cos\left(kL_p/2 \right) - i \frac{AL_p}{\sqrt{2\pi}} \left[\frac{\sin\left(kL_p/2 \right)}{\left(kL_p/2 \right)} \right] \sin\left(kL_p/2 \right)$$

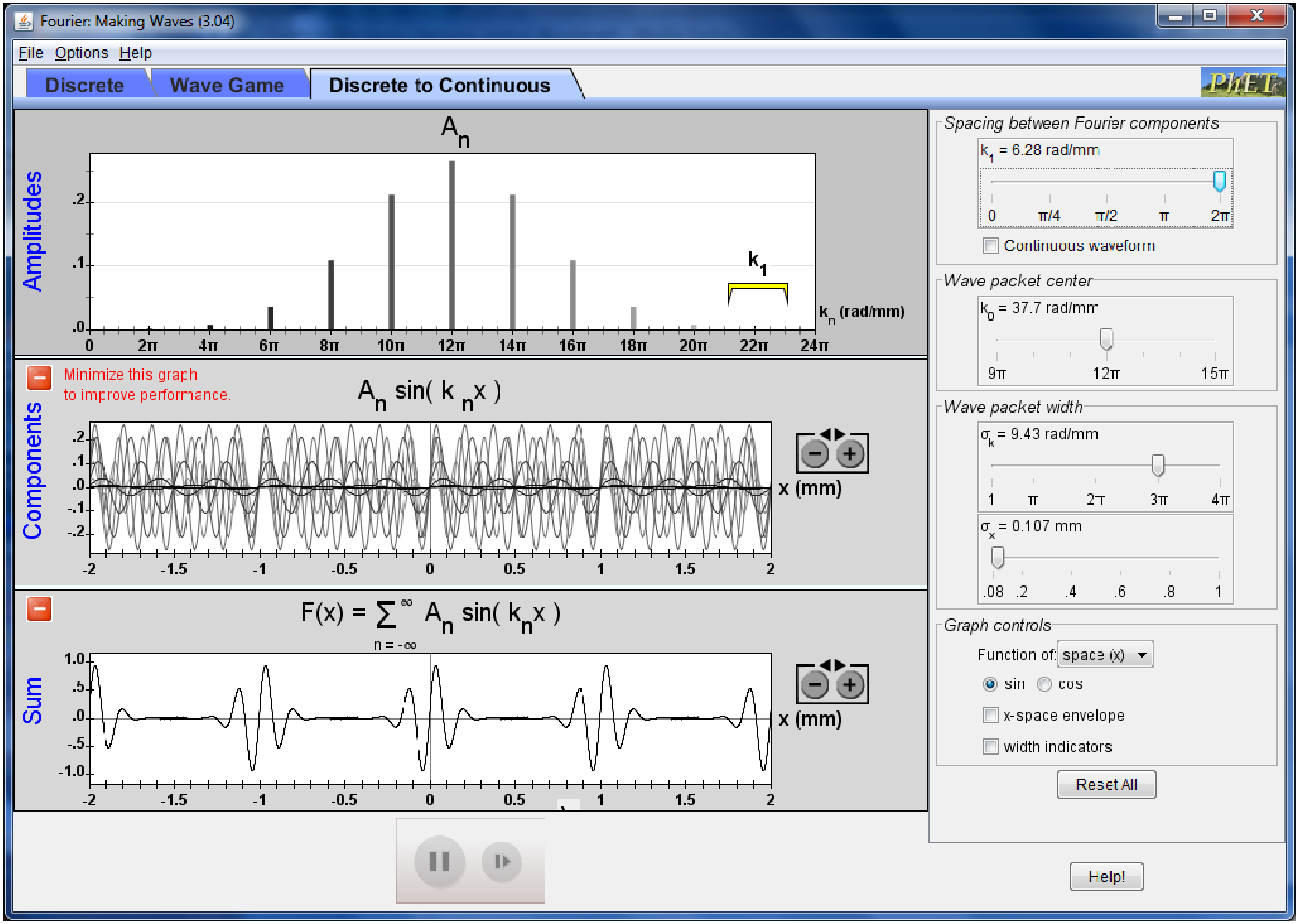
g(k) has real and imaginary parts

$$|g(k)|^2 = g^*(k)g(k) = \frac{A^2 L_p^{\ 2}}{2\pi} \left[\frac{\sin\left(kL_p/2 \right)}{\left(kL_p/2 \right)} \right]^2$$

magnitude of g(k)



Check it out!



http://phet.colorado.edu/simulations/sims.php?sim=Fourier_Making_Waves

Appendix: Using the Fourier Transform, you can now find the momentum eigenfunctions for the infinite square well

Application: Find the Momentum Eigenfunction for the n=3 quantum state in the Infinite Square Well

$$\Phi_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi_n(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} \Psi_n(x) e^{-ikx} dx$$

The Infinite Square Well - as an example

n	$\Psi_n(x)$	E_n	Levels
1	$\sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)$	$h^2/8m_e L^2$	$E_1=E_o$
2	$\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$	$4h^2/8m_e L^2$	$E_2=4E_o$
3	$\sqrt{\frac{2}{L}} \cos\left(\frac{3\pi x}{L}\right)$	$9h^2/8m_e L^2$	$E_3=9E_o$
4	$\sqrt{\frac{2}{L}} \sin\left(\frac{4\pi x}{L}\right)$	$16h^2/8m_e L^2$	$E_4=16E_o$

Choose n=3 for the Infinite Square Well – calculate the momentum eigenfunction

$$\Phi_3(k) = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} \Psi_3(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} \Psi_3(x) e^{-ikx} dx$$

$$\Psi_3(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{3\pi x}{L}\right)$$

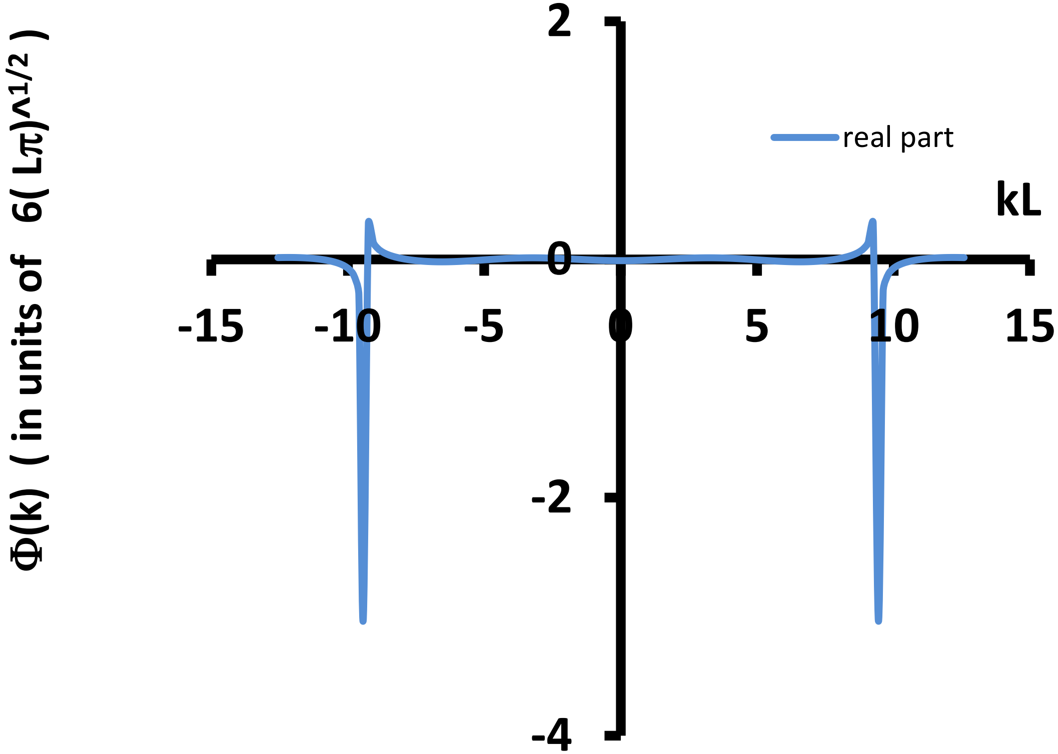
$$\Phi_3(k) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{L}} \int_{-L/2}^{+L/2} \cos\left(\frac{3\pi x}{L}\right) e^{-ikx} dx$$

$$= 6\sqrt{L\pi} \left[\left(\frac{1}{[kL - 3\pi][kL + 3\pi]} \right) \cos\left(\frac{kL}{2}\right) \right]$$

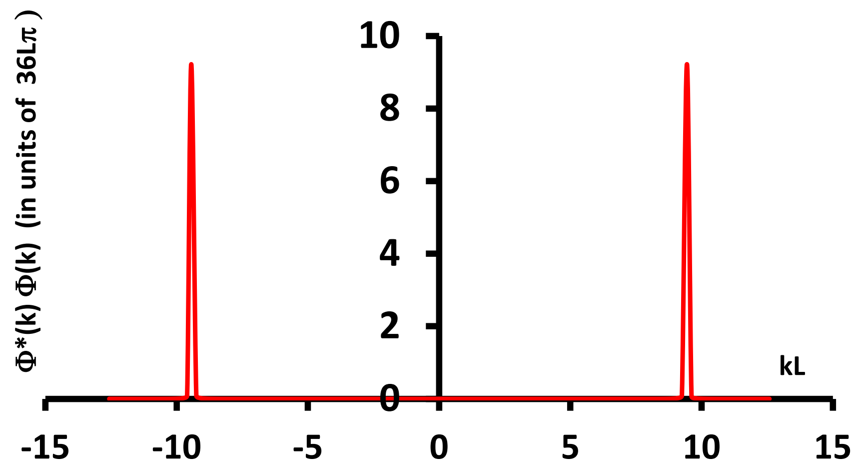
Note: math details included at end of Appendix

PLOTTING the RESULTS (n=3 for infinite square well)

Eigenfunction of momentum $\Phi(k)$



Probability $\Phi^*(k)\ \Phi(k)$
(sharp peaks at momentum eigenvalues)



The Math Details

$$\Phi_3(k) = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} \Psi_3(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} \Psi_3(x) e^{-ikx} dx$$

$$\Psi_3(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{3\pi x}{L}\right)$$

$$\Phi_3(k) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{L}} \int_{-L/2}^{+L/2} \cos\left(\frac{3\pi x}{L}\right) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{L}} \left[e^{-ikx} \left(\frac{-\left(\frac{3\pi}{L}\right)\sin\left(\frac{3\pi}{L}x\right) + ik\cos\left(\frac{3\pi}{L}x\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) \right]\Bigg|_{-L/2}^{+L/2}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{L}} \left[e^{-ikL/2} \left(\frac{-\left(\frac{3\pi}{L}\right)\sin\left(\frac{3\pi}{L}\frac{L}{2}\right) + ik\cos\left(\frac{3\pi}{L}\frac{L}{2}\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) - e^{-ik(-L/2)} \left(\frac{-\left(\frac{3\pi}{L}\right)\sin\left(\frac{3\pi}{L}\left(-\frac{L}{2}\right)\right) + ik\cos\left(\frac{3\pi}{L}\left(-\frac{L}{2}\right)\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{L}}\left[e^{-ik\frac{L}{2}} \left(\frac{-\left(\frac{3\pi}{L}\right)(-1) + ik \times 0}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) - e^{+ik(L/2)} \left(\frac{-\left(\frac{3\pi}{L}\right)(1) + ik \times 0}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{L}}\left[e^{-ik\frac{L}{2}} \left(\frac{\left(\frac{3\pi}{L}\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) + e^{+ik(L/2)} \left(\frac{\left(\frac{3\pi}{L}\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) \right]$$

$$= \frac{2}{\sqrt{L\pi}}\left[\left(\frac{\left(\frac{3\pi}{L}\right)}{k^2 - \left(\frac{3\pi}{L}\right)^2} \right) \cos\left(\frac{kL}{2}\right) \right]$$

$$= \frac{2}{\sqrt{L\pi}}\left(\frac{3\pi}{L}\right)\left[\left(\frac{1}{\left[k - \left(\frac{3\pi}{L}\right)\right]\left[k + \left(\frac{3\pi}{L}\right)\right]} \right) \cos\left(\frac{kL}{2}\right) \right]$$

$$= \frac{2}{\sqrt{L\pi}}\left(\frac{3\pi}{L}\right)\left[\left(\frac{L^2}{[kL - 3\pi][kL + 3\pi]} \right) \cos\left(\frac{kL}{2}\right) \right]$$

$$= \frac{6\pi L}{\sqrt{L\pi}}\left[\left(\frac{1}{[kL - 3\pi][kL + 3\pi]} \right) \cos\left(\frac{kL}{2}\right) \right]$$

$$= 6\sqrt{L\pi}\left[\left(\frac{1}{[kL - 3\pi][kL + 3\pi]} \right) \cos\left(\frac{kL}{2}\right) \right]$$