Modern Physics

Unit 4: Heisenberg’s Uncertainty Principle
Lecture 4.5: Wavepackets

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In this lecture, I assume you are familiar with the standard form for the Gaussian distribution function

\[ P(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} \quad \text{or} \quad P(k) = \frac{1}{\sqrt{2\pi\sigma_k^2}} e^{-\frac{(k-\bar{k})^2}{2\sigma_k^2}} \]

Only two parameters:
- Position: \( \bar{x} \)
- Width: \( \sigma_x \)
What we will show

If a particle (represented as a wave) has an uncertainty in position, then there must be a corresponding uncertainty in the momentum of that particle. You cannot measure both momentum and position of a quantum particle with high precision at the same time!

This is known as the Heisenberg Uncertainty Principle.

A guy is speeding down a highway in his car when he’s stopped by a police officer. “Do you know how fast you were going?” asks the officer. “No idea” answers the driver, “but I know exactly where I am.”

....as heard on the Bob and Tom radio show
From delocalized wavefunctions to wave packets

For the barrier transmission problems already discussed, the wave function $\Psi$ had a constant amplitude $A$ and extended from $x = -\infty$ to $x = +\infty$. These free particle states are completely delocalized.

$$\Psi = Ae^{ikx}$$

Q: How can we write a new wavefunction that is localized in space, to better approximate the motion of a particle?

A: Use the Fourier transform integral

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(k)e^{ikx} \, dk$$

where  

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x)e^{-ikx} \, dx$$
Write a wavefunction that localizes a particle near $x=0$

$$\Psi(x) = \left(\frac{1}{2\pi\alpha}\right)^{\frac{1}{4}} e^{i k_0 x} e^{-\frac{x^2}{4\alpha}}$$

why $\frac{1}{4}$ power?

plane wave

$e^{i k_0 x}$

envelope function

$e^{-\frac{x^2}{4\alpha}}$
**Important Consequences**

*IF*  
\[ \Psi(x) = \left( \frac{1}{2\pi\alpha} \right)^{\frac{1}{4}} e^{ik_0x} e^{-\frac{x^2}{4\alpha}} \]

*THEN*  
\[ \Psi^* \Psi = \sqrt{\frac{1}{2\pi\alpha}} e^{-\frac{x^2}{2\alpha}} \Rightarrow 2\alpha = 2\sigma_x^2 \Rightarrow \sigma_x = \sqrt{\alpha}; \quad \bar{x} = 0 \]

extent of wave \(\approx\) uncertainty in position

\[ 2\Delta x = 2\sqrt{\alpha} \]
Calculate $\Phi(k)$, the Fourier Transform Pair

$$
\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x) e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \frac{1}{2\pi\alpha} \right)^{1/4} e^{ik_{o}x} e^{-x^{2}/4\alpha} e^{-ikx} \, dx
$$

Wavefunction is

$$
\Phi(k) = \left( \frac{2\alpha}{\pi} \right)^{1/4} e^{-\alpha(k-k_{o})^{2}}
$$

(see Lecture 4.06 for details)

$$
\Phi^{*}(k) \Phi(k) = \sqrt{\frac{2\alpha}{\pi}} e^{-2\alpha(k-k_{o})^{2}} \Rightarrow 2\alpha = \frac{1}{2\sigma_{k}^{2}} \Rightarrow \sigma_{k} = \frac{1}{2\sqrt{\alpha}} ; \quad \bar{k} = k_{o}
$$

:. \quad \sigma_{x} \sigma_{k} \equiv \Delta x \Delta k = \sqrt{\alpha} \cdot \frac{1}{2\sqrt{\alpha}} = \frac{1}{2}

$$
p = \hbar k \Rightarrow \Delta p = \hbar \Delta k
$$

$$
\Delta x \Delta k = \Delta x \frac{\Delta p}{\hbar} = \frac{1}{2}
$$

$$
\Delta x \Delta p = \frac{\hbar}{2} = \frac{\hbar}{4\pi}
$$

Heisenberg Uncertainty Principle
Building Intuition - from a Fourier Series to a Fourier Integral

What to remember from this exercise?

Minimize this graph to improve performance.

Cannot show infinite number of components.

\[ F(x) = \int_{-\infty}^{+\infty} A(k) \sin(kx) \, dk \]
A Fundamental Property of Fourier Transform Pairs

\[ \Phi(k) \]

\[ \Psi(x) \]

\[ F(x) = \sum_{n=-\infty}^{\infty} A_n \sin(k_n x) \]

\[ \sigma_k = \pi \text{ rad} / \text{mm} \]
\[ \sigma_x = 0.317 \text{ mm} \]
\[ \sigma_k \sigma_x = 1 \text{ rad} \]