Modern Physics

Unit 8: Rules of Probability
Lecture 8.3: The Statistics of Large Numbers

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from Random Chance to Certainty

Keep track of all possible outcomes of N flips of a coin as N increases

For N=1: 1H or 1T
For N=2: 2H, 1T & 1H, 2T
For N=3: 3H, 2H & 1T; 1H & 2T; 3T
e tc.
Details for 16 flips of a Coin

<table>
<thead>
<tr>
<th>ALL Possible Outcomes</th>
<th>No. ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 H 16T</td>
<td>1</td>
</tr>
<tr>
<td>1H 15T</td>
<td>16</td>
</tr>
<tr>
<td>2H 14T</td>
<td>120</td>
</tr>
<tr>
<td>3H 13T</td>
<td>560</td>
</tr>
<tr>
<td>4H 12T</td>
<td>1820</td>
</tr>
<tr>
<td>5H 11T</td>
<td>4368</td>
</tr>
<tr>
<td>6H 10T</td>
<td>8008</td>
</tr>
<tr>
<td>7H 9T</td>
<td>11440</td>
</tr>
<tr>
<td>8H 8T</td>
<td>12870</td>
</tr>
<tr>
<td>9H 7T</td>
<td>11440</td>
</tr>
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</tr>
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<td>16H 0T</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ W = \frac{16!}{N_a ! N_b !} = \frac{16!}{N_a !(16-N_a)!} \]

*Note that the weight (or multiplicity, or degeneracy) plays a CRITICAL role in determining the probability of a particular outcome.*

\[ W_{3,13} = \frac{16!}{3!(13)!} = \frac{16 \cdot 15 \cdot 14}{3 \cdot 2} = 560 \]

\[ 2^{16} = 65,536 \text{ different microstates} \]

65,536
Q1: In 16 flips, what is the probability that you will obtain 8 heads?
A1: From table on previous slide: 12,870/65,536=19.6%
Q2: How certain are you of obtaining 8 heads?
A2: Measure 2σ!

Conclude that σ = 25% μ – significant spread compared to the mean
What happens as number of flips increases?

1. The mean number of heads is always \( \mu = \frac{N}{2} \).

2. The width (2\( \sigma \)) goes as \( 2\sqrt{N} \).

The ratio \( \frac{2\sigma}{\mu} \propto \frac{2\sqrt{N}}{N/2} \propto \frac{1}{\sqrt{N}} \to 0 \) as \( N \) becomes large.

As \( 2\sigma/\mu \) becomes smaller, the likelihood of obtaining a specified outcome becomes more certain, even though each individual event is still governed by random chance.
Transition from Random Chance to Certainty

Number of Flips

$2^7 = 128$ flips

$2^8 = 256$ flips

$2^{20} = 1,048,576$

$2^{30} = 1.07 \times 10^9$

$2^{40} = 1.1 \times 10^{12}$

$2^{50} = 1.1 \times 10^{15}$

Random chance

Virtual certainty

$2^{10} = 1024$

$1.07 \times 10^9$

$1.1 \times 10^{12}$

$1.1 \times 10^{15}$

one click
First, we will apply statistical ideas to a collection of molecules that forms an ideal gas.

http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html
Check out the ideal gas simulator at

Ideal Gas Assumptions - Microscopic Physics

1. A gas is made from a large number of molecules/atoms
2. The gas is comprised of identical molecules that are distinguishable
3. The size of individual molecules is very small compared to their average separation distance
4. Molecules obey Newton’s Laws of Motion
5. Molecules do not interact with each other
6. Collisions between molecules and container walls are elastic
7. Motion of molecules is entirely random

Assumptions 3 & 5 are most restrictive and limit the Ideal Gas model to low pressure (low density) gasses.
Historically, (Boyle, 1662; Charles, 1787; Gay-Lussac, 1802) it was known that for many common gasses

\[ PV \propto T \]

This proportionality can be converted to an equation (Clapeyeron, 1834):

\[ PV = Nk_B T \quad (N = \text{number of gas atoms}) \]

or

\[ PV = nRT \quad (n = \text{number of moles}) \]

This is an empirical law; there was no microscopic understanding of why it should be true.

\[ k_B = 1.38 \times 10^{-23} \text{ J/K}; \quad R = 8.314 \text{ J/K} = 0.082 \text{ liter atm mol}^{-1} \text{ K}^{-1} \]
Up Next - A Review of the Kinetic Theory of an Ideal Gas