Modern Physics

Unit 9: Statistical Laws of Nature
Lecture 9.7: Thermal de Broglie Wavelength

Ron Reifenberger
Professor of Physics
Purdue University
Important difference between classical and quantum statistics

In classical mechanics:
- identical objects can be distinguished
- two identical balls can be labeled (marked) so that they can be distinguished from each other
- the position of an object in a system of identical objects can be known at any given time

In quantum mechanics:
- identical particles cannot be labeled
- you know the probable position of two particles at some initial time
- you know the probable position of those two particles later in time
- cannot determine which particle went where
- this property is called Indistinguishability
For two identical particles, the wave nature of quantum mechanics prevents us from determining which of many possibilities actually occurred.

Separation comparable to the de Broglie wavelength.
1. When to use quantum statistics?

How to distinguish between a classical and quantum gas?

Volume = V
N = No. Particles
Mass = m

Classical:

\[ \frac{N}{V} = n \approx \frac{1 \text{ particle}}{\frac{4}{3} \pi s^3} = \frac{3}{4\pi s^3} \text{ (on average)} \]

Typically, for 1 mole at STP:

\[ n = \frac{N_A}{V} = \frac{6.02 \times 10^{23}}{22.4 \ell} \left( \frac{1 \ell}{0.001 m^3} \right) \]

\[ = 2.7 \times 10^{25} \text{ molecules / m}^3 = \frac{3}{4\pi s^3} \]

\[ \Rightarrow s = 2 \text{ nm} \]

Quantum:

\[ \lambda = \frac{h}{p} \Rightarrow E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \approx \frac{3}{2} k_B T \]

\[ \Rightarrow \lambda = \frac{h}{\sqrt{3mk_B T}} \]

“thermal” de Broglie wavelength
Expect classical statistics when $s >> \lambda$

Quantum gas (wavefunction overlap) \hspace{5cm} Classical gas (no wavefunction overlap)

$s << \lambda \quad s \approx \lambda \quad s >> \lambda$

from previous slide

\[
\frac{3}{2} k_B T = \frac{p^2}{2m} = \frac{1}{2m} \frac{h^2}{\lambda^2} \Rightarrow \lambda = \frac{h}{\sqrt{3mk_B T}}
\]

also, \[ s = \left( \frac{3}{4\pi n} \right)^{1/3} \]

when is $s >> \lambda$?

Typically, “$$>>$$” means a factor of 100

\[
\left( \frac{3}{4\pi n} \right)^{1/3} >> \frac{h}{\sqrt{3mk_B T}}
\]

\[
\left( \frac{3}{4\pi n} \right) >> \frac{h^3}{(3mk_B T)^{3/2}}
\]

when \[ n \left( \frac{N}{V} \right) << \frac{3}{4\pi} \left[ \frac{3mk_B T}{h^2} \right]^{3/2} \]

If TRUE, use classical statistics
If FALSE, use quantum statistics
Examples

1. Gas of $N_2$ molecules at temperature $T=300K$ and standard pressure. Mass of $N_2$ molecule is $4.7 \times 10^{-26}$ kg.

$$n = \frac{N}{V} \ll \frac{3}{4\pi} \left[ \frac{3mk_B T}{h^2} \right]^{3/2}$$

If TRUE, use classical statistics
If FALSE, use quantum statistics

$$I_s \quad 2.7 \times 10^{25} \ m^{-3} \ll \frac{3}{4\pi} \left[ \frac{3 \cdot 4.7 \times 10^{-26} \text{kg} \cdot 1.38 \times 10^{-23} \ J / K \cdot 300K}{(6.626 \times 10^{-34} \ J \cdot s)^2} \right]^{3/2}$$

$$2.7 \times 10^{25} \ m^{-3} \ll 0.239 \left[ \frac{5.84 \times 10^{-46} \ J \cdot kg}{4.39 \times 10^{-67} \ J^2 \cdot s^2} \right]^{3/2} = 0.239 \left[1.33 \times 10^{21} \ m^{-2} \right]^{3/2}$$

$$2.7 \times 10^{25} \ m^{-3} \ll 1.16 \times 10^{31} \ m^{-3} \quad \text{divide both sides by} \ 2.7 \times 10^{25}$$

$I_s \ 1 \ll 4.3 \times 10^5 \ ? \Rightarrow \text{classical statistics}$
2. Liquid He at temperature \( T=2 \) K. The molar mass of He is 4 grams. The mass of a He atom is about \( 6.6 \times 10^{-27} \) kg. The density of liquid He is 0.124 gm/cm\(^3\).

\[
n = \frac{N}{V} \ll \frac{3}{4\pi} \left[ \frac{3mk_B T}{h^2} \right]^{3/2}
\]

If TRUE, use classical statistics

If FALSE, use quantum statistics

\[
n_{\text{liquid He}} = \frac{N_A}{4 \text{ gm}} \cdot \left( 0.124 \text{ gm/cm}^3 \right) \cdot \left( \frac{100 \text{ cm}}{1 \text{ m}} \right)^3 = \frac{\left( 6 \times 10^{23} \right) (0.124) \left( 1 \times 10^6 \right)}{4} = 1.9 \times 10^{28} \text{ atoms/m}^3
\]

\[
Is \ 1.9 \times 10^{28} \text{ m}^{-3} \ll \frac{3}{4\pi} \left[ \frac{3 \times 6.6 \times 10^{-27} \text{ kg} \cdot 1.38 \times 10^{-23} \text{ J/K} \cdot 2K}{\left( 6.626 \times 10^{-34} \text{ J s} \right)^2} \right]^{3/2}
\]

\[
2.7 \times 10^{25} \text{ m}^{-3} \ll 0.239 \left[ \frac{5.5 \times 10^{-49} \text{ J kg}}{4.39 \times 10^{-67} \text{ J}^2 \text{ s}^2} \right]^{3/2} = 0.239 \left[ 1.25 \times 10^{18} \text{ m}^{-2} \right]^{3/2}
\]

\[
2.7 \times 10^{25} \text{ m}^{-3} \ll 3.4 \times 10^{26} \text{ m}^{-3} \quad (\text{divide both sides by } 2.7 \times 10^{25})
\]

\[
Is \ 1 \ll 12 \quad \Rightarrow \quad \text{use quantum statistics}
\]
Results for Different Substances

- $s \ll \lambda$ for low $T$ (quantum gas)
- $s \sim \lambda$ for $T$ near 10K
- $s \gg \lambda$ for all $T$ (classical gas)

\[ \lambda = \frac{h}{\sqrt{3mk_B T}} \]

Electron

- $H, He$
- $N_2$
- $Ar$

Increasing $T$

- $T=10K$
- $T=300K$
- $T=1000K$

$S$ (for typical ideal gas at STP)

$S$ (for typical liquid)

$S$ (for typical crystalline solid)

$\lambda$ for all $T$
Up Next - Quantum Statistics