Modern Physics

Unit 13: Special Relativity - Kinematics
Lecture 13.1: Length Contraction

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Measuring length using time

Tilt the Light Clock
(no mechanical parts)

When measured from a frame in which the clock is at rest

$t'=0$ emit light flash

If $L'=1.5\text{ m}$, then

$$\Delta t' = \frac{2L'}{c} = \frac{2(1.5\text{ m})}{3.0 \times 10^8 \text{ m/s}}$$

$$= 1.0 \times 10^{-8} \text{ s} = 10 \text{ ns}$$

Same result as before

One "tick" later: $\Delta t' = \frac{2L'}{c}$

Measure $\Delta t'$, infer $L'$
Measuring length using time
Clock now in motion. What do you see when viewed from stationary frame?

But \[ \Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v_f^2}{c^2}}} \] (time dilation) = \[ \frac{2L'}{c} \sqrt{\frac{1 - \frac{v_f^2}{c^2}}{1 - \frac{v_f^2}{c^2}}} \]

\[ L' = \frac{L}{\sqrt{1 - \frac{v_f^2}{c^2}}} = \gamma L \]

If \( L' = 1.5 \text{ m} \), then \( L = L' / \gamma \)

\[ \Delta t = \Delta t_1 + \Delta t_2 = \frac{2Lc}{c^2 - v_f^2} = \frac{2L}{c} \sqrt{\frac{1 - \frac{v_f^2}{c^2}}{1 - \frac{v_f^2}{c^2}}} \]

\[ t=0 \]
light flash emitted

\[ \Delta t_1 \]

\[ \Delta t_2 \]

\[ L + v_f \Delta t_1 = c \Delta t_1 \]
\[ L - v_f \Delta t_2 = c \Delta t_2 \]

one “tick” later
Length Contraction:

tilt the light clock by 90°

In Lecture 12.05, we established that when \( v_f = 0.866c \) (\( \gamma = 2.0 \)), a time interval for a moving clock was \( \frac{1}{2} \) of the time interval for a stationary clock.

If the clocks are rotated by 90°, what must happen to maintain this factor of \( \frac{1}{2} \)?

http://webphysics.davidson.edu/course_material/py230l/relativity/relativity-ex1.htm
Measuring the Length of an Object

Trust the equations. In general:

\[ x_2' = \gamma \left[ x_2 - v_f t_2 \right] \]
\[ x_1' = \gamma \left[ x_1 - v_f t_1 \right] \]
\[ x_2' - x_1' = \frac{(x_2 - x_1) - v_f (t_2 - t_1)}{\sqrt{1 - \beta^2}} \]

To measure length of rod from S, measure both ends at the same time, hence \( t_1 = t_2 \)

\[ x_2' - x_1' = \frac{(x_2 - x_1)}{\sqrt{1 - \beta^2}} \]
\[ \Delta x' = \gamma \Delta x \]
Tabulating Some Numbers

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v_f^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}; \quad \beta \equiv \frac{v_f}{c}
\]

\[
x_2 - x_1 = (x'_2 - x'_1) \sqrt{1 - \beta^2}
\]

Let \((x'_2 - x'_1) = 1 \text{ m}\)

<table>
<thead>
<tr>
<th>(\beta = \frac{v_f}{c})</th>
<th>0.100</th>
<th>0.300</th>
<th>0.600</th>
<th>0.800</th>
<th>0.900</th>
<th>0.950</th>
<th>0.990</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sqrt{1 - \beta^2})</td>
<td>0.995</td>
<td>0.954</td>
<td>0.800</td>
<td>0.600</td>
<td>0.436</td>
<td>0.312</td>
<td>0.141</td>
</tr>
<tr>
<td>(\gamma = \frac{1}{\sqrt{1 - \beta^2}})</td>
<td>1.005</td>
<td>1.048</td>
<td>1.250</td>
<td>1.667</td>
<td>2.294</td>
<td>3.205</td>
<td>7.092</td>
</tr>
<tr>
<td>(x_2 - x_1)</td>
<td>0.995 m</td>
<td>0.954 m</td>
<td>0.800 m</td>
<td>0.600 m</td>
<td>0.436 m</td>
<td>0.312 m</td>
<td>0.141 m</td>
</tr>
</tbody>
</table>

Length of rod measured from rest frame is always shorter than length of rod measured in moving frame!
Thinking it through, Step by Step

Transit time measured from $S$:
\[ \Delta t_1 = (\ell \rightarrow r) \quad \Delta t_2 = (r \rightarrow \ell) \]
\[ \Delta t_1 = \frac{\Delta x}{c - v_f} \quad \Delta t_2 = \frac{\Delta x}{c + v_f} \]
\[ \Delta t = \Delta t_1 + \Delta t_2 = 2 \frac{\Delta x}{c} \frac{1}{1 - \frac{v_f^2}{c^2}} = 2 \Delta x \gamma^2 \]
\[ \therefore \Delta x = \frac{1}{\gamma^2} \frac{c \Delta t}{2} \]

Transit time measured from $S'$:
\[ \Delta t' = 2 \frac{x_2' - x_1'}{c} = 2 \frac{\Delta x'}{c} \]

How are $\Delta t$ and $\Delta t'$ related?
We know that time dilation will occur: \[ \Delta t = \gamma \Delta t' \]

Summarizing

From S: \[ \Delta t = 2 \frac{\Delta x}{c} \gamma^2 \]
From S': \[ \Delta t' = 2 \frac{\Delta x'}{c} \]

\[ \Delta t = 2 \frac{\Delta x}{c} \gamma^2 = \gamma \Delta t' = \gamma \left[ 2 \frac{\Delta x'}{c} \right] \]

Since \( \gamma > 1 \), a logical consequence of time dilation is that the length of a moving object, when measured from a rest frame, is always shortened.
A More Intricate Example

Consider two arrows. The blue arrow is twice as long as the red arrow when at rest.

If the red arrow acquires a velocity of 0.5 c, what must be the velocity of the blue arrow so that its length is equal to the length of the red arrow when viewed by a stationary observer?

What we know:
- at rest (in $S'$) $\Delta x'_{blue} = 2\Delta x'_{red}$
- due to length contraction: $\gamma \Delta x = \Delta x' \Rightarrow \Delta x = \frac{\Delta x'}{\gamma}$

Measured when both are at rest

$v_{red} = 0.5c$

$v_{blue} = ?$

$$\Delta x = \frac{\Delta x'_{red}}{\gamma_{red}} = \frac{\Delta x'_{blue}}{\gamma_{blue}} = \frac{2\Delta x'_{red}}{\gamma_{blue}} \Rightarrow \frac{1}{\gamma_{red}} = \frac{2}{\gamma_{blue}}$$

$$\frac{1}{\gamma_{red}} = \sqrt{1 - \frac{v^2_{red}}{c^2}} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{2}{\gamma_{blue}}$$

$$\frac{4}{\gamma^2_{blue}} = \frac{3}{4} \Rightarrow \frac{1}{\gamma^2_{blue}} = \frac{3}{16} = \left(1 - \frac{v^2_{blue}}{c^2}\right)$$

$$\frac{v^2_{blue}}{c^2} = 1 - \frac{3}{16} \Rightarrow \frac{v^2_{blue}}{c^2} = \frac{13}{16} \Rightarrow v_{blue} = \frac{\sqrt{13}}{4} c$$

$v_{blue} = 0.9 c$
There is no change in length in a direction perpendicular to motion.

\[ x' = \gamma \left[ x - v_f t \right] \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma \left[ t - \frac{v_f x}{c^2} \right] \]
\[ u' = \frac{u - v_f}{1 - \left( \frac{v_f}{c^2} \right) u} \]
Hmmm..., Who is Really Correct?

Each measurement is CORRECT because relativity is really relative. There is NO preferred or favored reference frame.

Kyle’s observations are valid in Kyle’s frame

Cartman’s observations are valid in Cartman’s frame

There is no contradiction since it is physically impossible for both sets of measurements to refer to one and the same reference frame. There simply is no preferred frame of reference.
Up Next - Simultaneity Becomes Relative