ECE 255: L32

Low Frequency Response
(Sedra and Smith, 7th Ed., Sec. 10.1)

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Lundstrom: 2019
Announcements

HW10 Due 5:00 PM Friday, April 19 in EE-209 dropbox

LTSpice Project III Due 5:00 PM Wed, April 17
Outline

1) Decibels
2) High pass filter / STC circuits
3) LF response of CS
4) Design Example
5) LF Response of CE
6) Exact CS analysis
7) Summary
8) Appendix
A Bode plot is a graphical representation of frequency response data. It shows the magnitude and phase of a system's transfer function over a range of frequencies. The plot helps in understanding the system's behavior at different frequencies, including its gain bandwidth (BW) and unity gain frequency.

The plot includes the following elements:

- **DC coupled**:
  - The magnitude of the transfer function at DC is denoted as $|A_v|$.
  - The magnitude of the transfer function at the mid-frequency is denoted as $|A_v(mid)|$.

- **Lower “corner”**:
  - The frequency at which the magnitude of the transfer function is $|A_v(mid)|/\sqrt{2}$ is denoted as $f_L$.

- **Upper “corner”**:
  - The frequency at which the magnitude of the transfer function is $|A_v(mid)|/\sqrt{2}$ is denoted as $f_H$.

- **Unity gain frequency**:
  - The frequency at which the magnitude of the transfer function is $1$ is denoted as $A_i(f_T)| = 1$.

The Bode plot is a useful tool for engineers to analyze and design electrical circuits.
Decibels

**Power**

\[
A_p = \frac{P_{out}}{P_{in}} = \frac{v_o i_o}{v_i i_i} = \frac{v_o v_o}{v_i v_i} \frac{1}{R_L} = \left( \frac{v_o}{v_i} \right)^2 \frac{R_{in}}{R_L}
\]

\[
A_p \bigg|_{dB} = 10 \log_{10} \left( \frac{P_{out}}{P_{in}} \right)
\]

**Voltage**

\[
A_v = \frac{v_o}{v_i}
\]

\[
A_v \bigg|_{dB} = 20 \log_{10} \left( \frac{v_o}{v_i} \right)
\]

<table>
<thead>
<tr>
<th>Ap</th>
<th>Ap (dB)</th>
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<tbody>
<tr>
<td>0.01</td>
<td>-20 dB</td>
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<tr>
<td>0.1</td>
<td>-10 dB</td>
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<td>0.5</td>
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<td>100</td>
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<table>
<thead>
<tr>
<th>Av</th>
<th>Av (dB)</th>
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<tbody>
<tr>
<td>0.01</td>
<td>-40 dB</td>
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<tr>
<td>0.1</td>
<td>-20 dB</td>
</tr>
<tr>
<td>1/sqrt(2)</td>
<td>-3 dB</td>
</tr>
<tr>
<td>1</td>
<td>0 dB</td>
</tr>
<tr>
<td>10</td>
<td>20 dB</td>
</tr>
<tr>
<td>100</td>
<td>40 dB</td>
</tr>
</tbody>
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Bode plot

\[ |A_v| \text{ dB} \]

\[ |A_v(\text{mid})| \]

\(-3\text{ dB}\)

BW

lower 3 dB point

lower 3 dB point

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Single time constant (STC) circuit

\[ T(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} \]

\[ T(j\omega) = \frac{R_2}{R_1 + R_2 + 1/j\omega C} \]

\[ T(j\omega) = \frac{R_2/(R_1 + R_2)}{1 - j\omega C(R_1 + R_2)} \]

\[ \omega_L = \frac{1}{(R_1 + R_2)C} \]

\[ T(j\omega) = \frac{T_{mid}}{1 - j\omega L/\omega} \]
(STC) circuit: Bode plot

\[ T(j\omega) = \frac{T_{mid}}{1 - j\omega_L/\omega} \]

\[ |T(j\omega)| = \frac{T_{mid}}{\sqrt{1 + \frac{\omega_L^2}{\omega^2}}} \]

\[ \omega_L = \frac{1}{(R_1 + R_2)C} \]

\[ \omega >> \omega_L \quad |T(j\omega)| = T_{mid} \]

\[ \omega << \omega_L \quad |T(j\omega)| = T_{mid} \frac{\omega}{\omega_L} \]

20 dB /decade or 6 dB per octave
(STC) circuit: phase

\[ T(j\omega) = \frac{T_{mid}}{1 - j\omega_L/\omega} \]

\[ T(j\omega) = \frac{T_{mid}}{1 + \omega^2/\omega^2 (1 + j\omega_L/\omega)} \]

\[ \omega << \omega_L : \text{ phase } = 90^0 \]

\[ \omega = \omega_L : \text{ phase } = 45^0 \]

\[ \omega >> \omega_L : \text{ phase } = 0^0 \]
“Poles and zeros”

\[ T(j\omega) = \frac{T_{mid}}{1 + \omega_L/j\omega} \]

\[ s = j\omega \]

\[ T(s) = \frac{T_{mid}}{1 + \omega_L/s} \]

\[ T(s) = T_{mid} \times \frac{s}{s + \omega_L} \quad \text{STC has one pole} \]
The corner frequency (the pole) is one over a time constant – just find the RC time constant.

Kill the voltage source and find the Thevinen equivalent resistance that the capacitor sees.
Short cut

\[ R_{th} = R_1 + R_2 \]

\[ \omega_L = \frac{1}{R_{th}C} = \frac{1}{(R_1 + R_2)C} \]
Outline

1) Decibels
2) High pass filter / STC circuits
3) LF response of a discrete CS
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Now analyze this circuit
First corner frequency

\[ \omega_{L1} = \frac{1}{C_{C1} \left( R_{\text{series}} + R_{G1} \parallel R_{G2} \right)} \]

Expected be small (which is good)

**Important:** While we compute this corner frequency, we assume that the other C’s are shorted – this produces a STC circuit.
Second coupling capacitor

\[ V_{DD} \]

\[ R_{G1} \]

\[ C_{C1} \]

\[ R_{G2} \]

\[ R_{D} \]

\[ C_{C2} \]

\[ R_{S} \]

\[ C_{S} \]

\[ R_{L} \]

\[ R_{series} \]

\[ \nu_s \]
Second corner frequency

\[ \omega_{L2} = \frac{1}{C_{C2}(R_D + R_L)} \]

Expected to be reasonably small (which is also good)
Third coupling capacitor
Third corner frequency

\[ \omega_{LS} = \frac{1}{C_S \left( R_s \parallel \left( \frac{1}{g_m} \right) \right)} \]

Expected be the largest of the three.
Result

\[ \frac{V_o}{V_s} \approx A_{\text{mid}} \]

\[ \omega_L \approx \omega_{L_1} + \omega_{L_2} + \omega_{LS} \]

\[ \omega_L \approx \omega_{LS} \]
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Select the capacitors to:

- $R_{G1} = 10 \text{ meg}$
- $R_{series} = 100 \text{ k}$
- $R_{G1} = 9 \text{ meg}$
- $R_S = 1 \text{ k}$
- $V_{DD}$
- $R_D = 15 \text{ k}$
- $g_m = 1 \text{ mA/V}$
- $C_{C1}$
- $R_L = 15 \text{ k}$
- $C_{C2}$
- $C_S$

Make $f_L = 100 \text{ Hz}$
Bode plot

\[ |A_v| \text{ dB} \]

\[ |A_v(\text{mid})| = 288 \]

\[ f_L = 100 \text{ Hz} \]

\[ f_H \]

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Select the first capacitor

Make $f_L = 100$ Hz

$V_{DD}$

$R_{G1} = 10$ meg

$R_D = 15$ k

$C_{C2}$

$R_{series} = 100$ k

$C_{C1}$

$g_m = 1$ mA/V

$R_{G1} = 9$ meg

$R_S = 1$ k

$C_S$

$R_L = 15$ k

$v_s$
Source capacitance

$C_S$ is expected to be the largest of the three. It requires the largest $C$ to make the Corner Frequency spec.

$$\omega_{LS} = \frac{1}{C_S \left( R_S \parallel \left( 1/g_m \right) \right)} = 2\pi \times 100$$

$$\frac{1}{g_m} = 26 \, \Omega$$

Now make the other two corner frequencies 5-10 X smaller.

$$\frac{1}{C_S} = 628 \times 26$$

$$C_S = 62 \times 10^{-6} \, \text{F}$$
Select the second capacitors

Make $f_L = 100$ Hz

$R_{G1} = 10$ meg

$R_{G1} = 9$ meg

$R_{series} = 100$ k

$C_{C1}$

$C_{C2}$

$R_D = 15$ k

$g_m = 1$ mA/V

$C_S$

$R_L = 15$ k

$V_{DD}$

$R_S = 1$ k

$\nu_s$
The second capacitor

\[ \omega_{L1} = \frac{1}{C_{C1} \left( R_{\text{series}} + \frac{R_{G1}}{R_{G1} \parallel R_{G2}} \right)} = 2\pi \times 10 \quad \text{10 X smaller} \]

\[ C_{C1} = \left[ 62.8 \times (100 \times 10^3 + 4.7 \times 10^6) \right]^{-1} \]

\[ C_{C1} = 3.3 \times 10^{-9} \text{ F} \]
Select the third capacitor

\[ R_{G1} = 10 \text{ meg} \]

\[ R_{series} = 100 \text{ k} \]

\[ C_{C1} \]

\[ V_{DD} \]

\[ R_D = 15 \text{ k} \]

\[ \frac{g_m}{V} = 1 \text{ mA/V} \]

\[ C_{C2} \]

\[ R_L = 15 \text{ k} \]

\[ \nu_s \]

\[ R_{G1} = 9 \text{ meg} \]

\[ R_S = 1 \text{ k} \]

\[ C_S \]

Make \( f_L = 100 \text{ Hz} \)
The third capacitor

\[ \omega_{L2} = \frac{1}{C_{C2} \left( R_D + R_D \right)} = 2\pi \times 10 \]

10 X smaller

\[ C_{C2} = \left( 62.8 \times 30 \times 10^3 \right)^{-1} \]

\[ C_{C2} = 0.5 \times 10^{-6} \text{ F} \]
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Now analyze this circuit
Answers can be written down by inspection
$I_{C1} = 1.01 \text{ mA}$
$g_{m1} = 38.7 \text{ mS}$
$r_{\pi 1} = 5.24 \text{ k}$

$I_{C2} = 0.86 \text{ mA}$
$g_{m1} = 33.3 \text{ mS}$
$r_{\pi 1} = 6.17 \text{ k}$
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6) **Exact CS analysis**
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Our CS circuit

\[ V_{DD} \]

\[ R_{G1} \]

\[ C_{C1} \]

\[ R_{series} \]

\[ R_{D} \]

\[ C_{C2} \]

\[ R_{L} \]

\[ C_{S} \]

\[ v_s \]
1) Short two of the capacitors (they are supposed to have essentially zero impedance at the frequencies of interest).

2) Evaluate the STC circuit:

\[ \omega_{L1} = \frac{1}{C_1 R_{th1}} \]

3) Repeat for the other two capacitors.

4) Estimate the frequency response the amplifier as:

\[ |A_v(\omega)| = \frac{A_v(\text{mid})}{\sqrt{1 + (\omega_L/\omega)^2}} \]

\[ \omega_L = \omega_{L1} + \omega_{L2} + \omega_{L3} \]
Our (simplified) approach (ii)

This approach assume that the overall transfer function is the product of the three STC circuits:

\[ A_v(\omega) = A_v(\text{mid}) \left( \frac{1}{1 + \omega_{L1}/j\omega} \right) \left( \frac{1}{1 + \omega_{L2}/j\omega} \right) \left( \frac{1}{1 + \omega_{L3}/j\omega} \right) \]

\[ T(\omega) = T_1(\omega) \times T_2(\omega) \times T_3(\omega) \]

There is no guarantee that this will always work, but the procedure often gives us a good estimate for the corner frequency.

How would we solve the problem correctly?
Exact low frequency response of CS amplifier

\[ A_{v_s}(\omega) = A_{v_s}(\text{mid}) \left( 1 + \frac{\omega_z}{j\omega} \right) \left( 1 + \frac{\omega_{c1}}{j\omega} \right) \left( 1 + \frac{\omega_s}{j\omega} \right) \left( 1 + \frac{\omega_{c2}}{j\omega} \right) \]

(See Appendix)

Almost the product of 3 STC responses, but there is an additional zero.

\[ \omega_z = \frac{1}{R_s C_s} \ll \omega_s = \frac{1}{\left( R_s \parallel \frac{1}{g_m} \right) C_s} \]

So the zero is not important - it does not affect our conclusions about the LF corner frequency.
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Short-circuit time constant method

1) Short the other capacitors (they are supposed to have essentially zero impedance at the frequencies of interest).

2) Evaluate the STC circuit:

\[ \omega_{L1} = \frac{1}{C_1 R_{th1}} \]

3) Repeat for the other capacitors.

4) Select the largest corner frequency and estimate the frequency response the amplifier as:

\[ |A_v(\omega)| = \frac{A_v(\text{mid})}{\sqrt{1 + (\omega_L/\omega)^2}} \]

\[ \omega_L = \omega_{L1} + \omega_{L2} + \omega_{L3} \]
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Exact low frequency response of CS amplifier
Exact low frequency response of CS amplifier

\[ V_i(\omega) = \frac{R_G}{R_G + R_{series} + \frac{1}{j\omega C_{C1}}} V_s(\omega) \]

\[ V_i(\omega) = \left( \frac{R_G}{R_G + R_{series}} \right) \frac{1}{1 + \frac{\omega_{C1}}{j\omega}} V_s(\omega) \]

\[ \omega_{C1} = \frac{1}{(R_G + R_{series})C_{C1}} \]

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Exact low frequency response of CS amplifier
Exact low frequency response of CS amplifier

\[ V_{i}(\omega) \]

\[ V_{gs} \]

\[ V_{gs} = g_m V_{gs} \]

\[ I_d(\omega) \]

\[ V_{DD} \]

\[ R_D \]

\[ R_S \]

\[ C_S \]

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Exact low frequency response of CS amplifier

\[ I_d = g_m V_{gs} \]

\[ V_{gs} = V_i - I_d Z_s \]

\[ I_d = \frac{g_m V_i}{1 + g_m Z_s} \]

\[ I_d = g_m V_i \frac{Y_s}{Y_s + g_m} \]

\[ Y_s = \frac{1}{R_s} + j\omega C_s \]

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Exact low frequency response of CS amplifier

\[ I_d = g_m V_i \frac{Y_S}{Y_S + g_m} \quad Y_S = \frac{1}{R_S} + j\omega C_s \]

After some work, we find:

\[ I_d = g_m V_i \frac{1 + \frac{\omega_z}{\omega_s}}{1 + \frac{\omega_z}{\omega_s}} \quad \omega_s = \frac{1}{(R_S \parallel 1/g_m) C_s} \quad \omega_z = \frac{1}{R_s C_s} \]

In addition to the expected pole, we find a zero.
Exact low frequency response of CS amplifier
Putting it all together

\[ I_d(\omega) \]

\[ V_o(\omega) = I_{R_l}(\omega)R_L \]

\[ V_o(\omega) = -(R_D \ || R_L) \frac{1}{1 + \frac{\omega C_2}{j\omega}} I_d(\omega) \]

\[ I_{R_l}(\omega) = I_d(\omega) \frac{R_D}{R_D + \left( R_L + \frac{1}{j\omega C_2} \right)} \]

We find the expected pole.

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Exact low frequency response of CS amplifier

\[ V_o(\omega) = -\left( R_D \parallel R_L \right) \frac{1}{1 + \frac{\omega C_2}{j\omega}} I_d(\omega) \]

\[ I_d(\omega) = g_m \frac{1 + \frac{\omega Z}{j\omega}}{1 + \frac{\omega S}{j\omega}} V_i(\omega) \]

\[ V_i(\omega) = \left( \frac{R_G}{R_G + R_{\text{series}}} \right) \frac{1}{1 + \frac{\omega C_1}{j\omega}} V_s(\omega) \]

\[ \frac{V_o(\omega)}{V_i(\omega)} = A_{v_s}(\omega) = \left( \frac{R_G}{R_G + R_{\text{series}}} \right) g_m R_L \parallel R_D \times \left( 1 + \frac{\omega Z}{j\omega} \right) \left( 1 + \frac{\omega S}{j\omega} \right) \left( 1 + \frac{\omega C_2}{j\omega} \right) \]
Exact low frequency response of CS amplifier

\[ A_{v_s}(\omega) = A_{v_s}(\text{mid}) \frac{1+\frac{\omega_z}{j\omega}}{1+\frac{\omega_{c1}}{j\omega}} \times \frac{1+\frac{\omega_s}{j\omega}}{1+\frac{\omega_{c2}}{j\omega}} \]

\[ A_{v_s}(\text{mid}) = -\left(\frac{R_G}{R_G + R_{\text{series}}}\right) g_m \left( R_L \parallel R_D \right) \]

**Almost** the product of 3 STC responses, but there is an additional zero.

\[ \omega_z = \frac{1}{R_SC_S} \ll \omega_s = \frac{1}{\left( R_S \parallel 1/g_m \right)C_S} \]

So the zero is not important - it does not affect our conclusions about the LF corner frequency.