ECE 305: Spring 2018

Minority Carrier Diffusion Equation 1

Pierret, *Semiconductor Device Fundamentals* (SDF)
Chapter 3 (pp. 122-138)

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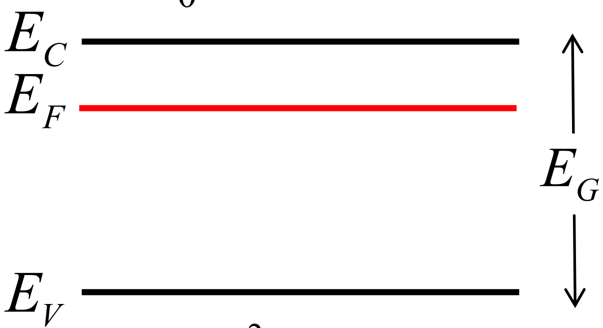
outline

1. Quasi-Fermi levels
2. Minority carrier diffusion equation
3. MCDE general solution procedure
4. Example #1: Steady-state illumination
5. Example #2: Transient, spatially uniform
6. Example #3: Steady-state, one-sided

where is the Fermi level?

Before we created the excess holes

$$n_0 = 10^{17} \ \text{cm}^{-3}$$



$$p_0 = \frac{n_i^2}{n_0} = 10^3 \ \text{cm}^{-3}$$

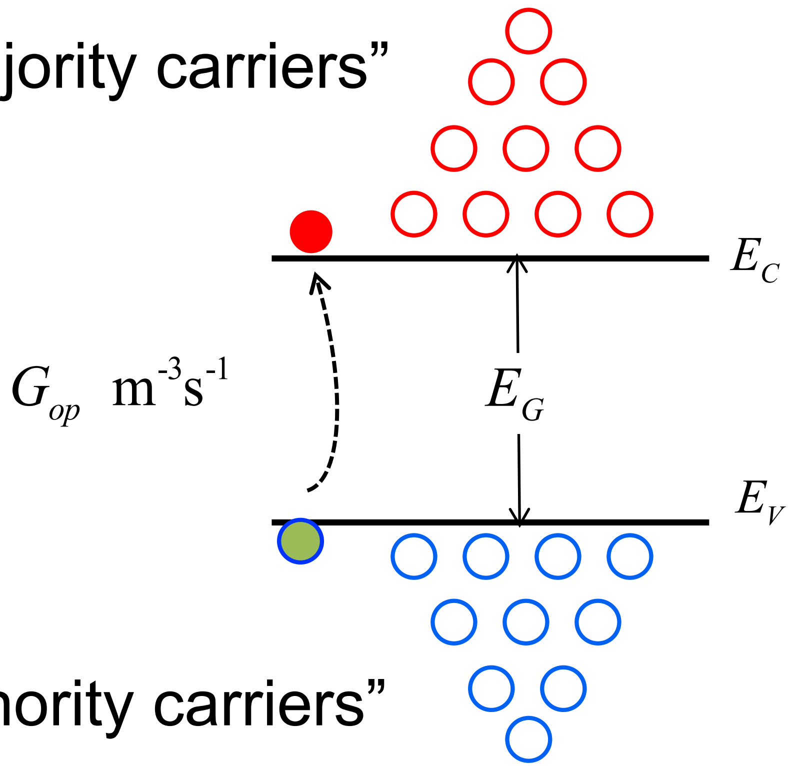
$$n_0 = n_i e^{(E_F - E_i)/k_B T}$$

$$p_0 = n_i e^{(E_i - E_F)/k_B T}$$

turn the light on: “excess carriers”

$$\Delta n = 10^{10} \ \ \text{cm}^{-3}$$

“majority carriers”



“minority carriers”

$$\Delta p = 10^{10} \ \ \text{cm}^{-3}$$

$$N_D = 10^{17} \ \ \text{cm}^{-3}$$

$$n_0 = 10^{17} \ \ \text{cm}^{-3}$$

$$n_0 p_0 = n_i^2$$

$$p_0 = \frac{n_i^2}{n_0} = 10^3 \ \ \text{cm}^{-3}$$

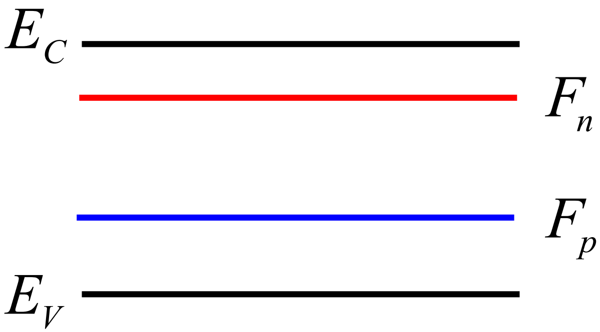
$$p = \Delta p \approx 10^{10} \ \ \text{cm}^{-3}$$

$$n = n_0 \approx 10^{17} \ \ \text{cm}^{-3}$$

“Low-level injection”

quasi-Fermi levels

$$n = 10^{17} \ \ \text{cm}^{-3}$$



$$p = 10^{10} \ \ \text{cm}^{-3}$$

$$n_0 = n_i e^{(E_F - E_i)/k_B T}$$

$$p_0 = n_i e^{(E_i - E_F)/k_B T}$$

$$n = n_i e^{(F_n - E_i)/k_B T}$$

$$F_n = E_F$$

$$p = n_i e^{(E_i - F_p)/k_B T}$$

$$F_p < E_F$$

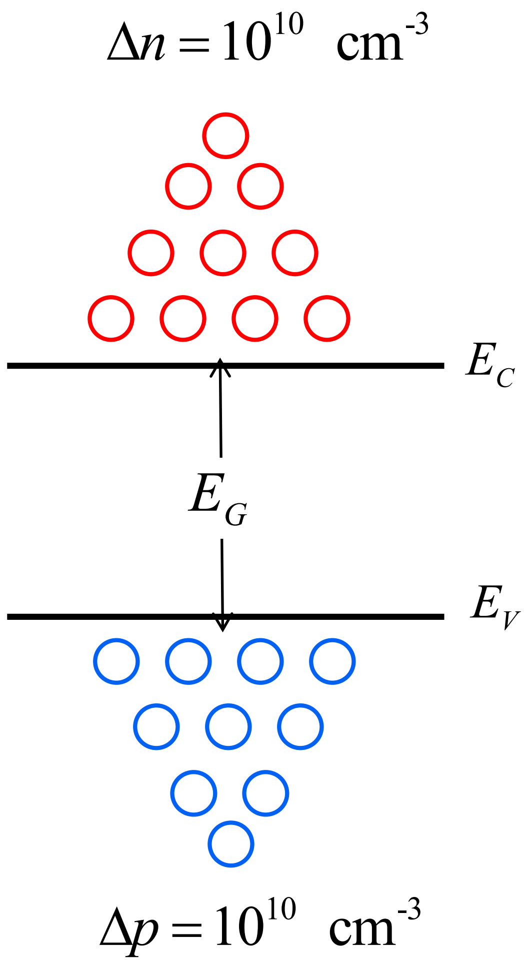
The QFL's are split

in equilibrium: $F_n = F_p = E_F$

equilibrium vs. non-equilibrium

equilibrium	non-equilibrium
$n_0 = n_i e^{(E_F - E_i)/k_B T}$	$n = n_i e^{(F_n - E_i)/k_B T}$
$p_0 = n_i e^{(E_i - E_F)/k_B T}$	$p = n_i e^{(E_i - F_p)/k_B T}$
$n_0 p_0 = n_i^2$	$np \neq n_i^2$
$f_0 = \dfrac{1}{1 + e^{(E - E_F)/k_B T}}$	$f_c = \dfrac{1}{1 + e^{(E - F_n)/k_B T}}$
	$1 - f_v = 1 - \dfrac{1}{1 + e^{(E - F_p)/k_B T}}$

turn the light off



Question: What happens?

Answer: The system returns to equilibrium.

How long does it take?
A time known as the "minority carrier lifetime". τ_p sec

$$R_p = \left.\frac{\partial p}{\partial t}\right|_{R-G} = -\frac{\Delta p}{\tau_p}$$

(low-level injection)

current and QFL's

$$\vec{J}_p = pq\mu_p\vec{\mathcal{E}} - qD_p\vec{\nabla}p = p\mu_p\vec{\nabla}\left(F_p/q\right)$$

$$\vec{J}_n = nq\mu_n\vec{\mathcal{E}} + qD_n\vec{\nabla}n = n\mu_n\vec{\nabla}\left(F_n/q\right)$$

$$p = n_i e^{\left(E_i - F_p\right)/k_B T}$$

$$n = n_i e^{\left(F_n - E_i\right)/k_B T}$$

$$\frac{dp}{dx} = n_i e^{\left(E_i - F_p\right)/k_B T} \times \frac{1}{k_B T}\left(\frac{dE_i}{dx} - \frac{dF_p}{dx}\right) = \frac{p}{k_B T}\left(\frac{dE_i}{dx} - \frac{dF_p}{dx}\right)$$

$$\frac{dE_i}{dx} = q\mathcal{E}_x$$

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Minority Carrier Diffusion Equation

$$\nabla \bullet D = q\left(p - n + N_D^+ - N_A^-\right)$$

$$\rightarrow q\left(p_0 + \Delta p - n_0 - \Delta p + N_D^+ - N_A^-\right) \rightarrow 0$$

$$\mathbf{J}_N = qn\mu_N \boldsymbol{\mathcal{E}} + qD_N \nabla n$$

$$\rightarrow qD_N \frac{\partial n}{\partial x} \qquad (\boldsymbol{\mathcal{E}} \sim 0)$$

$$\frac{\partial n}{\partial t} = \frac{1}{q}\nabla \bullet \mathbf{J}_N - r_N + g_N \rightarrow \frac{\partial\left(n_0 + \Delta n_p\right)}{\partial t} = \frac{1}{q}\frac{\partial \mathbf{J}_N}{\partial x} - \frac{\Delta n_p}{\tau_n} + g_N$$

$$\frac{\partial\left(n_0 + \Delta n_p\right)}{\partial t} = \frac{\partial\left(\Delta n_p\right)}{\partial t} = D_N \frac{\partial^2 \Delta n_p}{\partial x^2} - \frac{\Delta n_p}{\tau_n} + g_p$$

minority carrier diffusion equation

$$\frac{\partial p}{\partial t} = -\nabla \bullet \left(\frac{\vec{J}_p}{q} \right) + G_p - R_p$$

(hole continuity equation)

$$\frac{\partial p}{\partial t} = -\frac{d}{dx}\left(\frac{J_p}{q} \right) + G_L - R_p$$

(1D, generation by light)

$$\frac{\partial \Delta p}{\partial t} = -\frac{d}{dx}\left(\frac{-qD_p \, d\Delta p/dx}{q} \right) + G_L - \frac{\Delta p}{\tau_p}$$

(low-level injection, no electric field)

$$\frac{\partial \Delta p}{\partial t} = D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$$

(D_p spatially uniform)

Various approximations …

Time dependence

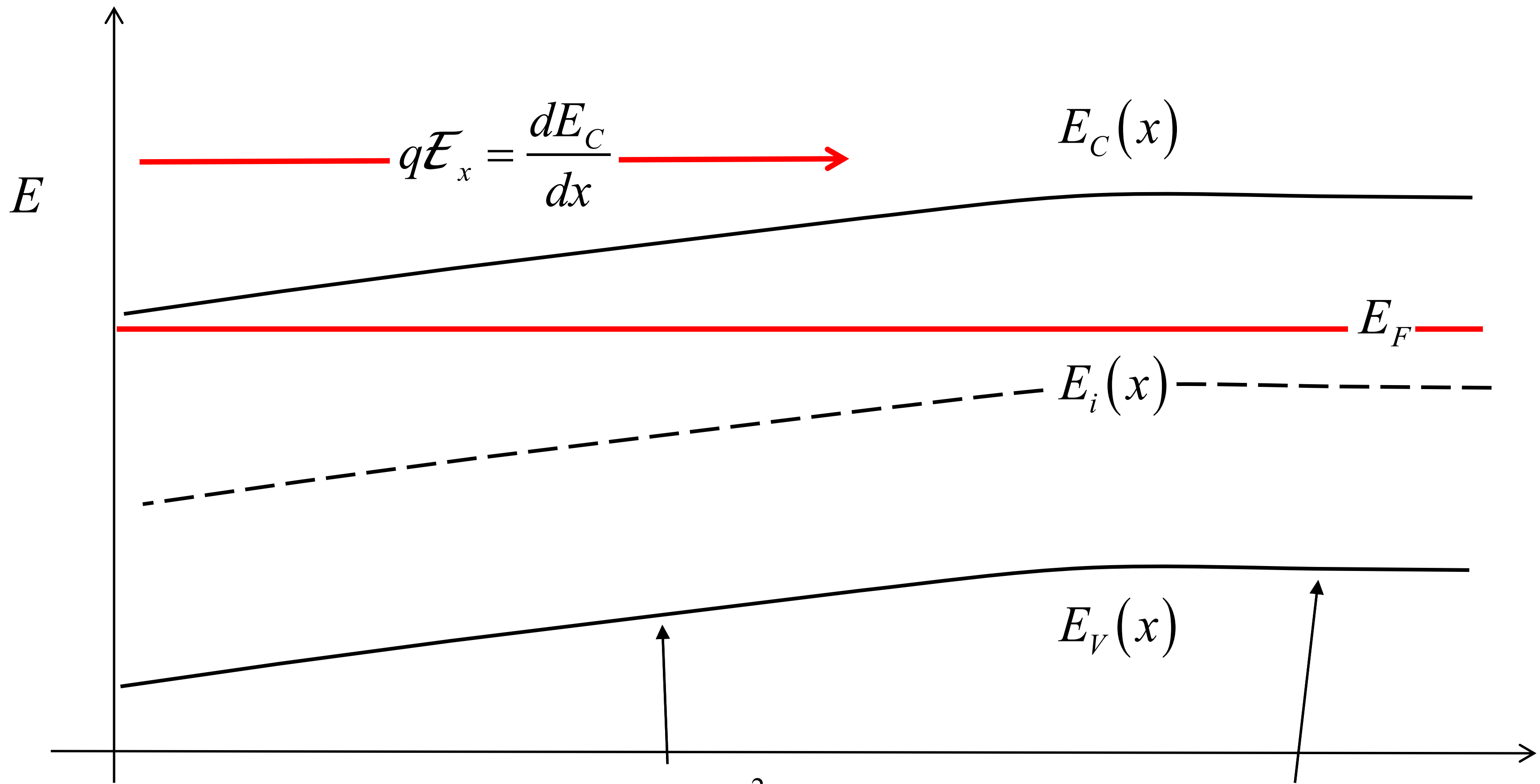
recombination

generation

$$\frac{\partial\left(\Delta n_p\right)}{\partial t} = D_N \frac{\partial^2 \Delta n_p}{\partial x^2} - \frac{\Delta n_p}{\tau_n} + g_p$$

density gradient

e-band diagram



$$\frac{\partial \Delta p}{\partial t} \neq D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$$

MCDE only valid here

outline

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How to solve (some) Exam 2 problems

Step 1: From material information (semiconductor, doping, etc.), calculate carrier densities, Fermi level, etc. Start with the majority carriers, $p = N_A - N_D, \ \ or\ n = N_D - N_A$. Then get the other carrier from $np = n_i^2$

Step 2: Use band-diagram to calculate potential profile, electric field, $E = dE_c/dx$, or $E = -dV/dx$, and $\kappa_s \epsilon_0\, \rho = dE/dx$, etc. For homogenous semiconductor with a battery attached, $E = V_B/L$.

Step 3: Decide if this is drift-related problem (resistivity, velocity, mobility, etc.), or a diffusion related problem (light turning on-off, etc.)

Step 4A: For a drift-problem use $\rho^{-1} = qn\mu_p E \ + \ qp\mu_p E$. For μ, you may be given a number, or table, or diffusion coefficient, etc. Learn how to read such a table.

Step 4B: For a diffusion problem, read carefully for clues to simplify the minority carrier equation.

How to solve equations

Step 4B: Two general types of minority diffusion problem.

i) Determine if electron or the hole is the minority carrier.

ii) If holes are the minority carriers, write the equation:

$$\frac{\partial \Delta p}{\partial t} = D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$$

iii) If steady-state, drop the time-derivative. If transient, keep the time derivative. If spatially uniform, drop the diffusion term. Without light, drop the generation term. If the region is very short, drop the recombination term. Choose the solutions from the following table.

iv) Use the boundary conditions to complete solution.

How to solve equations

$$\frac{\partial \Delta p}{\partial t} = D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$$

Transient

$$\frac{\partial \Delta p}{\partial t} = G_L - \frac{\Delta p}{\tau_p},$$

solution

$$\Delta p = G_L \tau_p + B\, e^{-\frac{t}{\tau_p}}$$

Boundary condition for B:
Concentration before light was turned on?

Steady State

$$0 = D_p \frac{d^2 \Delta p}{dx^2} + G_L - \frac{\Delta p}{\tau_p}$$

solution

$$\Delta p = Ae^{-\frac{x}{L_D}} + Be^{+\frac{x}{L_D}} + G_L \tau_p$$

If $L \ll L_D$,

$$\Delta p = A + Bx + G_L \tau_p$$

BC to determine A and B:
Concentration at leftmost and rightmost points

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Example #1: Reading the problem

For all the following problems, assume p-type silicon at room temperature, uniformly doped with $N_A = 10^{17}$ cm^{-3}, $\mu_n = 300$ cm^2/V sec, $\tau_n = 10^{-6}$ s. From these numbers, we find:

$$D_n = \frac{k_B T}{q}\mu_n = 7.8 \text{ cm}^2/\text{s} \qquad L_n = \sqrt{D_n \tau_n} = 27.9\ \mu\text{m}$$

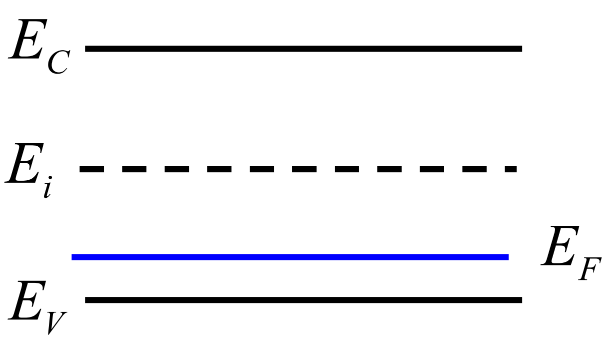
Step 1: Determine carrier densities.

a) The majority carrier is ?
b) the majority carrier concentration is?
c) The minority carrier concentration is?
d) How did they get the diffusion coefficient?
e) Special words: uniformly doped.

Example #1: determining minority carriers

P-type / **in** equilibrium

$$n_0 = \frac{n_i^2}{p_0} = 10^3 \ \text{ cm}^{-3}$$



$$p_0 = 10^{17} \ \text{ cm}^{-3}$$

$$p_0 = n_i e^{(E_i - E_F)/k_B T}$$

$$10^{17} = 10^{10} e^{(E_i - E_F)/k_B T}$$

$$E_F = E_i - 0.41 \text{ eV}$$

example #1: P-type sample in uniform injection

Steady-state, uniform generation, no spatial variation

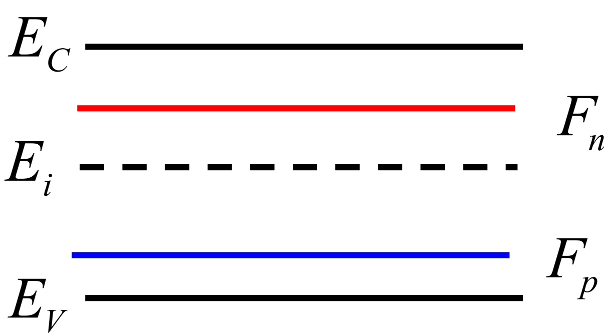
Solve for Δn and for the QFL's.

1) Simplify the MCDE
2) Solve the MCDE
3) Deduce F_n from Δn

Example #1: Solution

P-type / **out of** equilibrium

$$n_0 = \frac{n_i^2}{p_0} = 10^3 \ \ \text{cm}^{-3}$$



$$p_0 = 10^{17} \ \ \text{cm}^{-3}$$

$$F_p = E_i - 0.41 \text{ eV}$$

$$n \approx \Delta n = n_i e^{(F_n - E_i)/k_B T}$$

$$10^{14} = 10^{10} e^{(F_n - E_i)/k_B T}$$

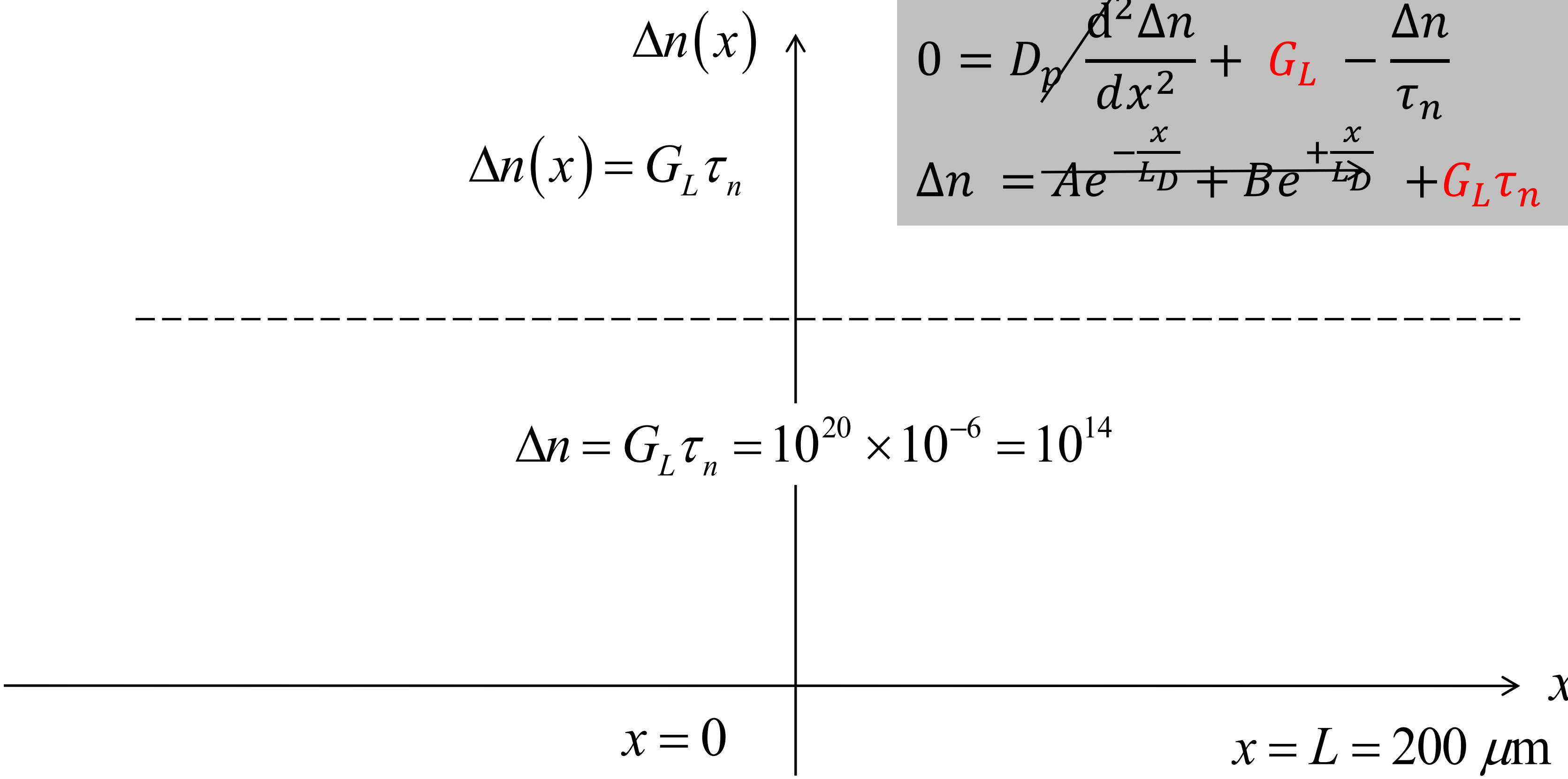
$$F_n = E_i + 0.24 \text{ eV}$$

Steady-state, uniform generation, no spatial variation

Example #1: Solution

$$0 = \cancel{D_p \frac{d^2 \Delta n}{dx^2}} + G_L - \frac{\Delta n}{\tau_n}$$

$$\Delta n = \cancel{A e^{-\frac{x}{L_D}} + B e^{+\frac{x}{L_D}}} + G_L \tau_n$$



Steady-state, uniform generation, no spatial variation

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Example #2

Now turn off the light.
Transient, no generation, no spatial variation

iii) Which approximate equation should I choose?

$$\frac{\partial \Delta n}{\partial t} = G_L - \frac{\Delta n}{\tau_p},$$

solution

$$\Delta n = G_L \tau_n + B\, e^{-\frac{t}{\tau_p}}$$

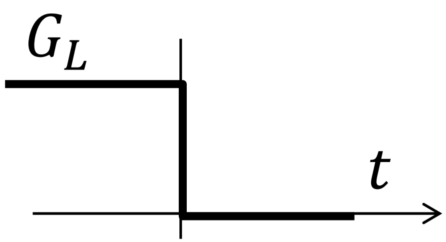
$$0 = D_p \frac{d^2 \Delta n}{dx^2} + G_L - \frac{\Delta p}{\tau_p}$$

Solution:

$$\Delta p = A e^{-\frac{x}{L_D}} + B e^{+\frac{x}{L_D}} + G_L \tau_n$$

iv) What is my boundary condition:
Just before, light was turned on for a long-time, before it is turned off.

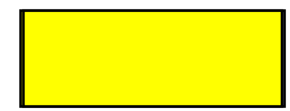
$$\Delta n(0^-) = G_L\, \tau_n = \Delta n(0^+)$$



Example 2A: Transient, **No Illumination**

(uniform)

$$\frac{\partial n}{\partial t} = \cancel{\frac{1}{q}\nabla \bullet \mathbf{J}_N} - r_N + g_N \qquad \mathbf{J}_N = qn\mu_N E + qD_N \nabla n$$

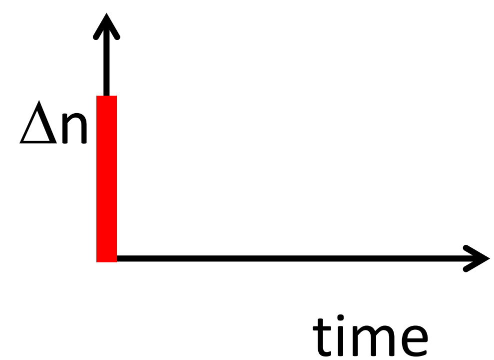


$$\frac{\partial(\cancel{n_0} + \Delta n)}{\partial t} = -\frac{\Delta n}{\tau_n}$$

Acceptor doped

(uniform)

$$\frac{\partial p}{\partial t} = \cancel{\frac{-1}{q}\nabla \bullet \mathbf{J}_p} - r_P + g_p \qquad \mathbf{J}_p = qp\mu_p E - qD_P \nabla p$$



$$\frac{\partial(\cancel{p_0} + \Delta p)}{\partial t} = -\frac{\Delta p}{\tau_p}$$

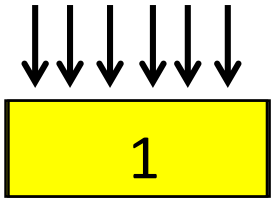
Majority carrier

$$\nabla \bullet D = q\left(p - n + N_D^+ - N_A^-\right) = q\left(p_0 + {\color{red}\Delta n} - n_0 - {\color{blue}\Delta p} + N_D^+ - N_A^-\right) = 0$$

Example 2B: Transient, **Uniform Illumination**

(uniform)

$$\frac{\partial n}{\partial t} = \frac{1}{q}\cancel{\nabla \bullet \mathbf{J}_N} - r_N + g_N \qquad \mathbf{J}_N = qn\mu_N E + qD_N \nabla n$$



$$\frac{\partial(\cancel{n_0} + \Delta n)}{\partial t} = -\frac{\Delta n}{\tau_n} + G$$

Acceptor doped

(uniform)

$$\frac{\partial p}{\partial t} = \frac{-1}{q}\cancel{\nabla \bullet \mathbf{J}_p} - r_P + g_p \qquad \mathbf{J}_p = qp\mu_p E - qD_P \nabla p$$

$$\frac{\partial(\cancel{p_0} + \Delta p)}{\partial t} = -\frac{\Delta p}{\tau_p} + G$$

Majority carrier

$$\nabla \bullet D = q\left(p - n + N_D^+ - N_A^-\right) = q\left(p_0 + \Delta n - n_0 - \Delta p + N_D^+ - N_A^-\right) = 0$$

Example 2 Summary

Analytical solutions

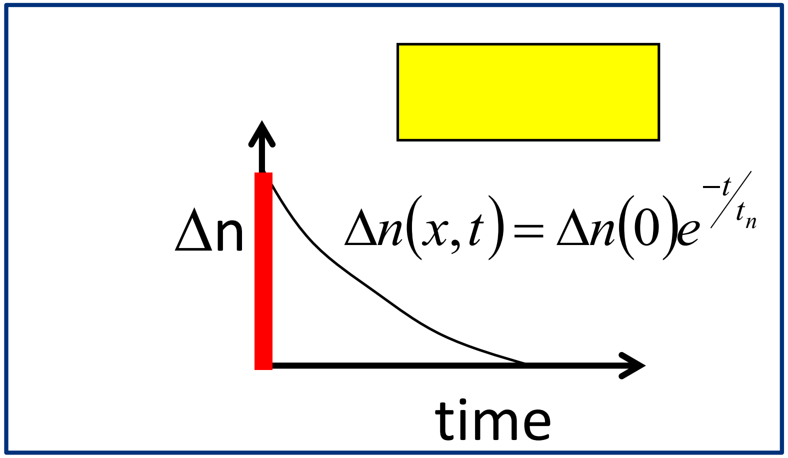
$$\frac{\partial n}{\partial t} = \frac{1}{q}\nabla \bullet \mathbf{J}_N - r_N + g_N$$

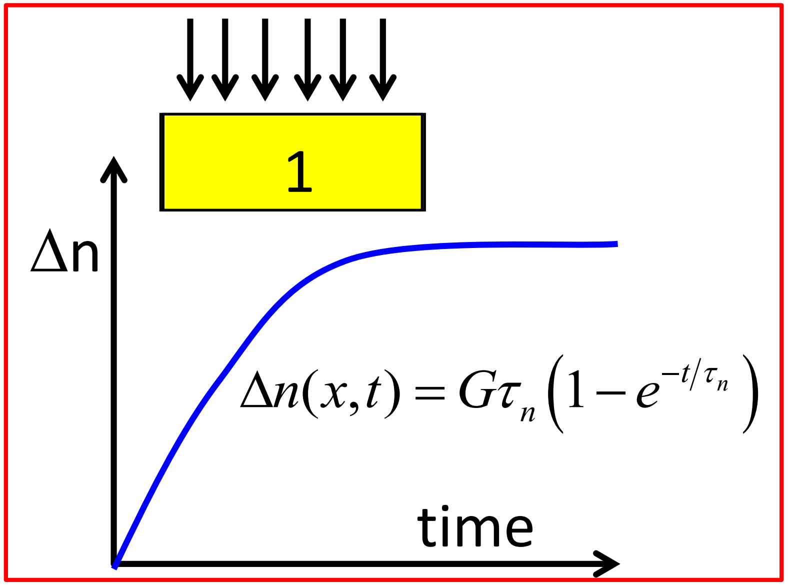
$$\mathbf{J}_N = qn\mu_N \mathcal{E} + qD_N \nabla n$$

$$\frac{\partial p}{\partial t} = \frac{-1}{q}\nabla \bullet \mathbf{J}_P - r_P + g_P$$

$$\mathbf{J}_P = qp\mu_P \mathcal{E} - qD_P \nabla p$$

$$\nabla \bullet D = q\left(p - n + N_D^+ - N_A^-\right)$$





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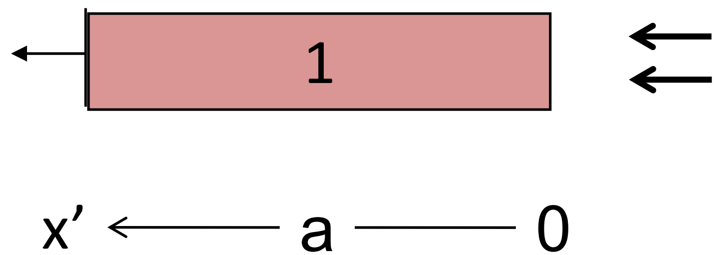
Example #3: One-sided Carrier Diffusion

Steady state, no generation/recombination

$\Delta n(x',t) = 0$

Metal contact



$$\cancel{\frac{\partial n}{\partial t}} = \frac{1}{q}\frac{dJ_n}{dx} - \cancel{r_N} + \cancel{g_N}$$

$$\mathbf{J}_N = qn\cancel{\mu_N} E + qD_N \frac{dn}{dx}$$

$$0 = D_N \frac{d^2 n}{dx^2}$$

Example 3B: One-sided carrier diffusion Long sample (steady state)

Steady-state, sample **long** compared to the diffusion length.
i.e., a short diffusion length

$\Delta p\left(x=0\right)$ fixed

1) Simplify the MCDE
2) Solve the MCDE
3) Deduce F_p from Δp

$$\frac{\partial \Delta p}{\partial t} = D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$$

Example 3B: One-sided carrier diffusion Long sample (steady state)

Steady-state, sample **long** compared to the diffusion length. No generation. $\Delta n(x=0)=10^{12}\ \text{cm}^{-3}$ fixed

Step 3) What type of problems are we talking about?

Step 4B) Key words: steady-state, without light, long device

ii) If I write $\frac{\partial \Delta p}{\partial t} = D_p \frac{d^2 \Delta p}{dx^2} - \frac{\Delta p}{\tau_p} + G_L$ for MCDE, would I be right?

iii) Which approximate equation should I choose:

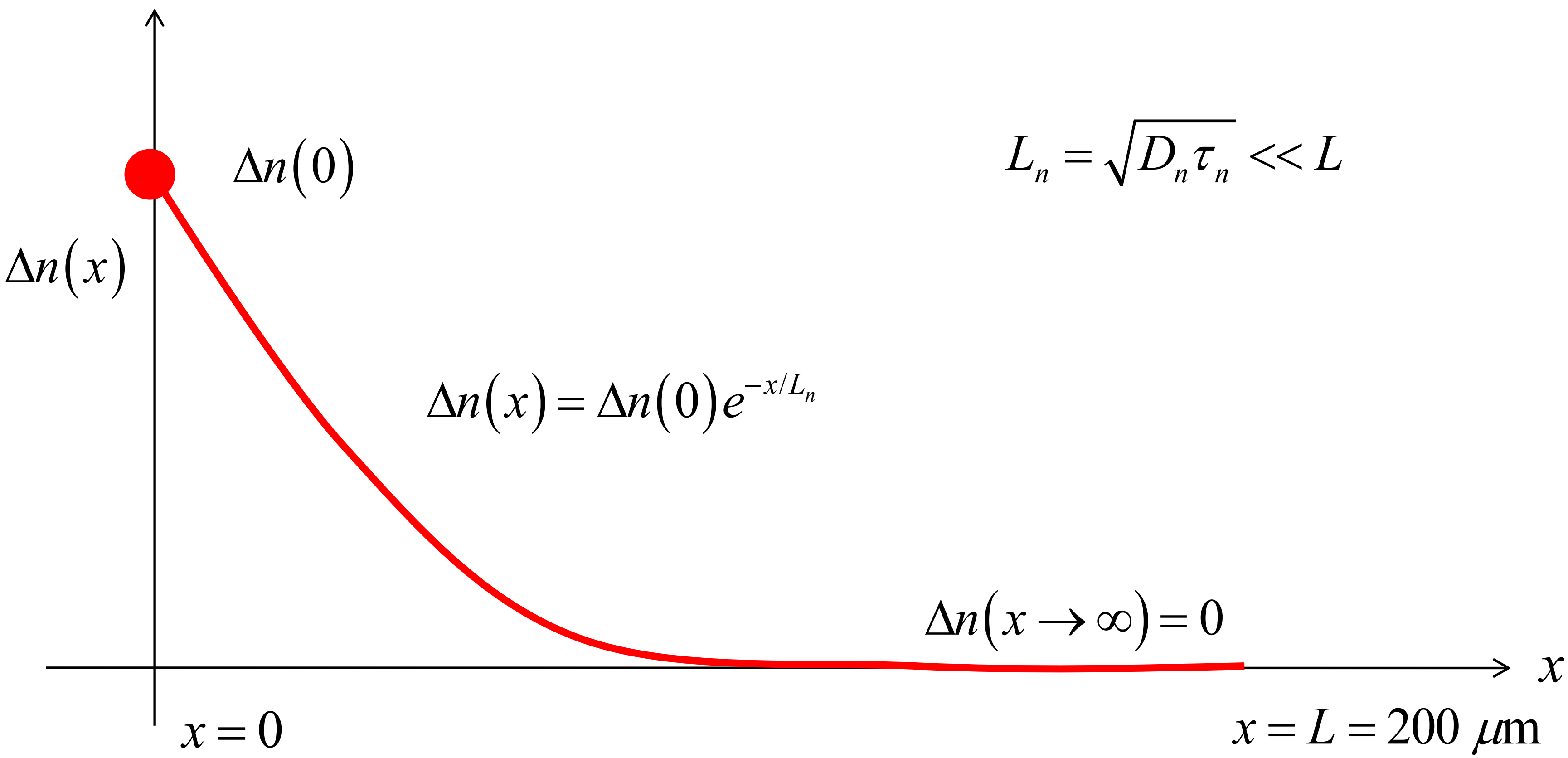
$$\frac{\partial \Delta n}{\partial t} = G_L - \frac{\Delta n}{\tau_p},$$

solution $\Delta n = G_L \tau_n + B\, e^{-\frac{t}{\tau_p}}$

$$0 = D_p \frac{\mathrm{d}^2 \Delta n}{dx^2} + G_L - \frac{\Delta p}{\tau_p}$$

solution $\Delta p = Ae^{-\frac{x}{L_D}} + Be^{+\frac{x}{L_D}} + G_L \tau_n$

Example 3B: One-sided carrier diffusion Long sample (steady state)

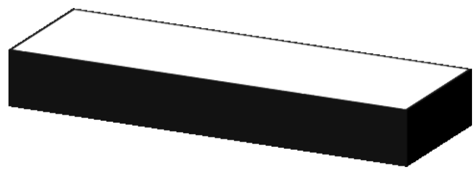


Steady-state, sample **long** compared to the diffusion length.

Example 3B: One-sided carrier diffusion Long sample (steady state)

Steady-state, sample is **5 micron long**. No generation.

$\Delta n(x=0)=10^{12}\ \text{cm}^{-3}$



$\Delta n(x=5\ \mu\text{m})=0$

Which approximate equation should I choose:

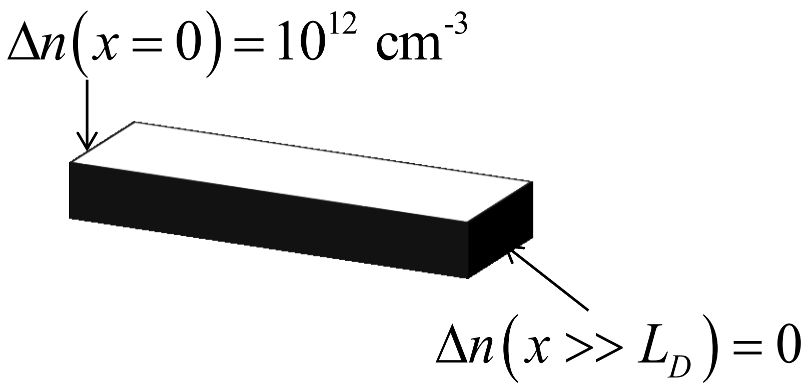
$\frac{\partial \Delta n}{\partial t}=G_L-\frac{\Delta n}{\tau_p}$, **solution** $\Delta n=G_L\tau_n+B\,e^{-\frac{t}{\tau_p}}$

$0=D_p\frac{d^2\Delta n}{dx^2}+G_L-\frac{\Delta p}{\tau_p}$ **solution** $\Delta p=Ae^{-\frac{x}{L_D}}+Be^{+\frac{x}{L_D}}+\cancel{G_L\tau_n}$

or $\Delta p=A+Bx+\cancel{G_L\tau_n}$ $L\ll L_D$

What are my boundary conditions?

Example 3B: One-sided carrier diffusion Long sample (steady state)



$$\Delta n(x) = Ax + B$$

$$\Delta n(x=0) = 10^{12}\ \text{cm}^{-3}$$

$$\Delta n(x=L) = 0$$

$$\Delta n(x) = \Delta n(0)\left(1 - \frac{x}{L}\right)$$

Example 3C: One-sided carrier diffusion Intermediate sample (steady state)

Steady-state, sample is **30 micrometers long**. No generation.

$\Delta n(x=0) = 10^{12}\ \text{cm}^{-3}$ fixed

$\Delta n(x = 30\ \mu\text{m}) = 0$

$$\frac{\partial \Delta n}{\partial t} = D_p \frac{d^2 \Delta n}{dx^2} - \frac{\Delta}{\tau_n} + G_L$$

$$0 = D_p \frac{d^2 \Delta n}{dx^2} - \frac{\Delta}{\tau_n} + 0$$

$$\frac{d^2 \Delta n}{dx^2} - \frac{\Delta n}{L_n} = 0 \qquad L_n \square \sqrt{D_n \tau_n}$$

1) Simplify the MCDE
2) Solve the MCDE
3) Deduce F_p from Δp

$L_n = 28\ \mu\text{m}$ $\qquad$ $L = 30\ \mu\text{m}$

Example 3C: One-sided carrier diffusion Intermediate sample (steady state)

Steady-state, sample is **30 micrometers long**. No generation.

$$\Delta n(x=0) = 10^{12}\ \text{cm}^{-3}$$ fixed

$$\Delta n(x = 30\ \mu\text{m}) = 0$$

$$\frac{d^2 \Delta n}{dx^2} - \frac{\Delta n}{L_n} = 0$$

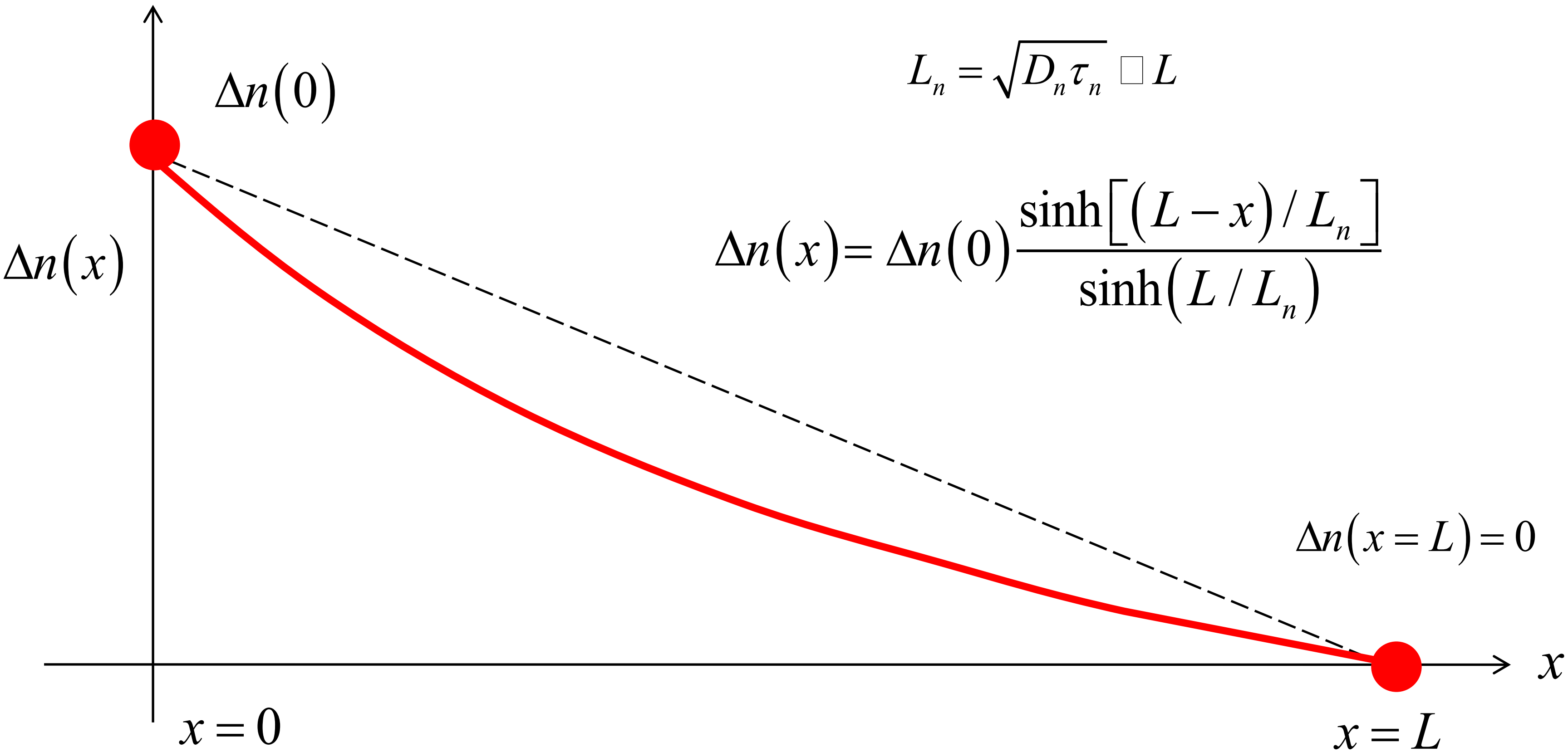
$$\Delta n(x) = Ae^{-x/L_n} + Be^{+x/L_n}$$

$$\Delta n(0) = A + B = 10^{12}$$

$$\Delta n(L) = Ae^{-L/L_n} + Be^{+L/L_n} = 0$$

1) Simplify the MCDE
2) Solve the MCDE
3) Deduce F_p from Δp

Example 3C: One-sided carrier diffusion Intermediate sample (steady state)



$$L_n = \sqrt{D_n \tau_n} \; □ \; L$$

$$\Delta n(x) = \Delta n(0) \frac{\sinh\left[(L - x)/L_n\right]}{\sinh(L/L_n)}$$

Steady-state, sample neither long nor short compared to the diffusion length.

Example #3 summary: One-sided carrier diffusion

Length scale	Solution type
Long ($L \gg L_D$)	Decaying exponentials
Short ($L \ll L_D$)	Linear
Intermediate ($L \sim L_D$)	Hyperbolic functions

<u>Notes:</u>

L is length of region where MCDE applies

$L_D = \sqrt{D_{n,p}\tau_{n,p}}$ is the diffusion length for carriers

conclusions

- The continuity equation $$\frac{\partial n}{\partial t} = -\nabla \bullet \left(\frac{\vec{J}_n}{-q} \right) + G_n - R_n$$ ensures consistency of current & carrier conc.
- Recombination and generation caused by multiple processes, which can be described simply
- Quasi-Fermi level splitting is associated with the resulting deviations from equilibrium
- We will often be using minority carrier diffusion equation to understand the mechanics of carrier transport in electronic devices. Review the problem carefully to see if the assumption of minority carrier transport is satisfied.
- Divide all complex problems into solvable parts, solve the parts sequentially and then put the partial solutions back by using proper boundary conditions to arrive at the complete solution.