SOLUTIONS: ECE 656 Homework (Week 12)

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1) Show that the following relation (eqn. (4.119) in *Fundamentals of Carrier Transport*) is true:

$$e^{-\frac{\pi}{\rho_S}R_{MN,OP}} + e^{-\frac{\pi}{\rho_S}R_{NO,PM}} = 1$$

Solution:

From slide 6 in Lecture 18, Fall 2011 (or eqns. (4.117) and (4.118) of FCT):

$$R_{MN,OP} = \frac{\rho_S}{\pi}\ln\left(\frac{(a+b)(b+c)}{b(a+b+c)}\right) \rightarrow \frac{\pi R_{MN,OP}}{\rho_S} = \ln\left(\frac{(a+b)(b+c)}{b(a+b+c)}\right) \qquad \text{(i)}$$

$$R_{NO,PM} = \frac{\rho_S}{\pi}\ln\left(\frac{(a+b)(b+c)}{ac}\right) \rightarrow \frac{\pi R_{NO,PM}}{\rho_S} = \ln\left(\frac{(a+b)(b+c)}{ac}\right) \qquad \text{(ii)}$$

From (i) and (iii)

$$e^{-\frac{\pi}{\rho_S}R_{MN,OP}} + e^{-\frac{\pi}{\rho_S}R_{NO,PM}} = \frac{b(a+b+c)) + ac}{(a+b)(b+c)} = \frac{ab+b^2+bc+ac}{ab+ac+b^2+bc} = 1$$

$$\boxed{e^{-\frac{\pi}{\rho_S}R_{MN,OP}} + e^{-\frac{\pi}{\rho_S}R_{NO,PM}} = 1}$$

2) For n-type, bulk silicon doped at $N_D = 10^{17}$ cm-3 the room temperature mobility is 800 cm2/V-s. Answer the following questions. Some potentially useful information is:

$N_C = 3.23\times 10^{19}$ cm^{-3} $\quad N_V = 1.83\times 10^{19}$ cm^{-3} $\quad E_G = 1.11\,\text{eV}$ $\quad \upsilon_T = 1.05\times 10^7$ cm/s

2a) Estimate the mean-free-path for backscattering. Make reasonable assumptions, but clearly state them.

Solution:

Since $n_0 = N_D << N_C$, the semiconductor is nondegenerate. For a nondegenerate semiconductor,

$$D_n = \frac{k_B T}{q}\mu_n = \frac{\upsilon_T \langle\langle\lambda\rangle\rangle}{2}$$

$$\langle\langle\lambda\rangle\rangle = \frac{2k_B T}{q}\frac{\mu_n}{\upsilon_T} = 2(0.026)\frac{800}{1.05\times10^7} = 4.0\times10^{-6}\text{ cm}$$

$$\boxed{\langle\langle\lambda\rangle\rangle = 40\text{ nm}}$$

2b) Estimate the Seebeck coefficient. Make reasonable assumptions, but clearly state them.

Solution:

$$S = -\left(\frac{k_B}{q}\right)\left[\frac{\Delta_n}{k_B T_L} - \eta_F\right]$$

$$n_0 = N_C e^{\eta_F} \rightarrow \eta_F = \ln(n_0/N_C) = -5.78 \quad \text{(non-degenerate semiconductor)}$$

Assume $\Delta_n = 2k_B T_L$ (non-degenerate, constant mfp)

$$S = -(86)[2 + 5.78] = -669\ \mu\text{V/K}$$

$$\boxed{S = -669\ \mu\text{V/K}}$$

2c) Estimate the number of conduction channels per cm². Make reasonable assumptions, but clearly state them.

Solution:

$$\sigma = n_0 q\mu_n = \frac{2q^2}{h}\langle M\rangle\langle\langle\lambda\rangle\rangle \rightarrow \langle M\rangle = \frac{h}{2q^2}\frac{\sigma}{\langle\langle\lambda\rangle\rangle}$$

$$\sigma = n_0 q\mu_n = 12.8\text{ S/cm}$$

$$\langle M\rangle = \frac{h}{2q^2}\frac{\sigma}{\langle\langle\lambda\rangle\rangle} = 13\times10^3\,\frac{12.8}{40\times10^{-7}} = 4.2\times10^{10}\text{ cm}^{-2}$$

$$\boxed{\langle M\rangle = 4.2\times10^{10}\text{ cm}^{-2}}$$

3) It is tempting to estimate the momentum relaxation time, $\langle\langle \tau_m \rangle\rangle$, from the mobility and then to multiply by a velocity to get the mean-free-path. Give the correct expression for the mfp for backscattering in 2D – in terms of $\langle\langle \tau_m \rangle\rangle$ as extracted from the measured mobility. You may assume a non-degenerate semiconductor.

Solution:

$$D_n = \frac{\upsilon_T \langle\langle \lambda \rangle\rangle}{2} = \frac{k_B T}{q}\mu_n = \frac{k_B T}{q}\frac{q\langle\langle \tau_m \rangle\rangle}{m^*} \tag{i}$$

$$\langle\langle \lambda \rangle\rangle = \frac{2k_B T}{\upsilon_T}\frac{\langle\langle \tau_m \rangle\rangle}{m^*} = \frac{2k_B T}{m^*}\sqrt{\frac{\pi m^*}{2k_B T}}\langle\langle \tau_m \rangle\rangle = \sqrt{\frac{2\pi k_B T}{m^*}}\langle\langle \tau_m \rangle\rangle \tag{ii}$$

$$\langle\langle \lambda \rangle\rangle = \frac{\pi}{\pi}\sqrt{\frac{2\pi k_B T}{m^*}}\langle\langle \tau_m \rangle\rangle = \pi\sqrt{\frac{2k_B T}{\pi m^*}}\langle\langle \tau_m \rangle\rangle \tag{iii}$$

$$\boxed{\langle\langle \lambda \rangle\rangle = \pi\,\upsilon_T \langle\langle \tau_m \rangle\rangle}$$

The above expression relates the mean-free-path for backscattering (the "Landauer mean-free-path") to the transport average scattering time.

4) The purpose of this homework assignment is to solve the Boltzmann Transport Equation for a particle with charge $+Zq$, where Z is an integer > 1. This may occur in problems like the flow of ions through channels in cell walls or the flow of ions inside a battery.

4a) Solve the BTE in the relaxation time approximation assuming a constant relaxation time, and a small electric field, but no concentration gradient. Use the result to derive an equation for the drift current.

Solution:

BTE:

$$\frac{\partial f}{\partial t} + \upsilon_x \frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p_x} = -\frac{\delta f}{\tau_m} \tag{i}$$

steady-state, spatially uniform, constant scattering time:

$$F\frac{\partial f}{\partial p_x} = -\frac{\delta f}{\tau_0} \tag{ii}$$

$$\delta f = -\tau_0 F \frac{\partial f}{\partial p_x} \tag{iii}$$

$$F = +Zq\mathcal{E}_x \tag{iv}$$

$$\delta f = -\left(\tau_0 Zq \frac{\partial f}{\partial p_x}\right)\mathcal{E}_x \tag{v}$$

$$f = f_0 + \delta f = f_0 - \left(\tau_0 Zq \frac{\partial f}{\partial p_x}\right)\mathcal{E}_x \tag{vi}$$

This is a displaced Maxwellian with the displacement in momentum being:

$$p_{dx} = \left(\tau_0 Zq\right)\mathcal{E}_x \tag{vii}$$

The average drift velocity is

$$\left\langle \upsilon_x \right\rangle = \frac{p_{dx}}{m^*} = \left(\frac{q\tau_0 Z}{m^*}\right)\mathcal{E}_x = \mu\mathcal{E}_x, \tag{viii}$$

so the mobility of these charged particles is:

$$\mu = Z\left(\frac{q\tau_0}{m^*}\right), \tag{ix}$$

and the drift current is

$$J_{nx} = nZq\left\langle \upsilon_x \right\rangle = nZq\mu\mathcal{E}_x \tag{x}$$

$$\boxed{\begin{aligned} J_{nx} &= n\left(Zq\right)\mu\mathcal{E}_x \\ \mu &= Z\left(\frac{q\tau_0}{m^*}\right) \end{aligned}}$$

4b) Solve the BTE in the relaxation time approximation assuming a constant relaxation time, and a small concentration gradient, but no electric field.

Solution:

In this case, the steady-state BTE becomes:

$$\upsilon_x \frac{\partial f}{\partial x} = -\frac{\delta f}{\tau_0}$$

$$\boxed{\delta f = -\tau_0 \upsilon_x \frac{\partial f_0}{\partial x}} \qquad \text{(xi)}$$

4c) Use the result from 4b) to derive an equation for the diffusion current.

Solution:

$$J_{nx} = \frac{1}{\Omega}\sum_{\vec{k}} (Zq)\upsilon_x \delta f = \frac{1}{\Omega}\sum_{\vec{k}} (Zq)\upsilon_x \left(-\tau_0 \upsilon_x \frac{\partial f_0}{\partial x}\right)$$

$$J_{nx} = -(Zq)\tau_0 \frac{1}{\Omega}\sum_{\vec{k}} \upsilon_x^2 \frac{\partial f_0}{\partial x} = -(Zq)\tau_0 \frac{d}{dx}\left(\frac{1}{\Omega}\sum_{\vec{k}} \upsilon_x^2 f_0\right)$$

$\upsilon_x^2 \to \upsilon^2/3$ (spherical symmetry)

$$J_{nx} = -\frac{1}{3}(Zq)\tau_0 \frac{d}{dx}\left(\frac{1}{\Omega}\sum_{\vec{k}} \upsilon^2 f_0\right) = -\frac{2}{3}\frac{(Zq)\tau_0}{m^*}\frac{d}{dx}\left(\frac{1}{\Omega}\sum_{\vec{k}} \frac{m^*\upsilon^2}{2} f_0\right)$$

We can recognize:

$$\frac{1}{\Omega}\sum_{\vec{k}} \frac{m^*\upsilon^2}{2} f_0 = n\left(\frac{3}{2}k_B T\right),$$

so the drift current is (assume constant temperature)

$$J_{nx} = -k_B T \frac{(Zq)\tau_0}{m^*}\frac{dn}{dx} = -(Zq)D_n \frac{dn}{dx} \qquad \text{(xii)}$$

where

$$D_n = k_B T \frac{\tau_0}{m^*} = \frac{k_B T}{q}\left(\frac{q\tau_0}{m^*}\right) = \frac{k_B T}{q}\left(\frac{\mu}{Z}\right) \qquad \text{(xiii)}$$

$$\boxed{\begin{aligned} J_{nx} &= -(Zq)D_n \frac{dn}{dx} \\ D_n &= \frac{k_B T}{q}\left(\frac{q\tau_0}{m^*}\right) \end{aligned}}$$

4d) Find the Einstein relation for these charged particles.

Solution:

Using (ix) and (xiii)

$$\frac{D}{\mu} = \frac{D_n = k_B T \tau_0 / m^*}{Z\left(q\tau_0 / m^*\right)} = \frac{k_B T}{Zq}$$

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$$\boxed{\frac{D}{\mu} = \frac{k_B T}{Zq}}$$

5) A Hall effect experiment is performed on a n-type semiconductor with a length of 2.65 cm, a width of 1.70 cm, and a thickness of 0.0520 cm, in a magnetic field of 0.5 T. The current in the sample along its length is 200 μA. The potential difference along the length of the sample is 195 mV and across the width is 21.4 mV.

5a) What is the carrier concentration of the sample?

Solution:

Recall that in 2D:

$$\vec{\mathcal{E}} = \rho_S \vec{J}_n + \left(\rho_S \mu_n r_H\right) \vec{J}_n \times \vec{B}$$

$$\mathcal{E}_y = \rho_S J_y - \left(\rho_S \mu_n r_H\right) J_x B_z = -\left(\rho_S \mu_n r_H\right) J_x B_z$$

$$V_H = -W\mathcal{E}_y = \left(\rho_S \mu_n r_H\right) I_x B_z = \left(\frac{1}{n_S q \mu_n} \mu_n r_H\right) I_x B_z = \left(\frac{r_H}{q n_S}\right) I_x B_z$$

which is the same as eqn. (4.110) in FCT

$$V_H = \frac{r_H}{q n_S} B_z I$$

$$\frac{n_S}{r_H} = \frac{B_z I}{qV_H} = \frac{0.5\times 200\times 10^{-6}}{1.6\times 10^{-19}\left(21.4\times 10^{-3}\right)} = 2.92\times 10^{16}\,\text{cm}^{-2}$$

$$\frac{n_S/r_H}{t} = n_H = \frac{2.92\times 10^{16}\,\text{cm}^{-2}}{0.0520\ \ \text{cm}} = 5.62\times 10^{17}\,\text{cm}^{-3}$$

$$\boxed{n_H = \frac{n}{r_H} = 5.62\times 10^{17}\,\text{cm}^{-3}}$$

Not the carrier concentration, but the "Hall carrier concentration."

5b) What is the mobility?

Solution:

$$R_{xx} = \frac{195\times 10^{-3}}{200\times 10^{-6}} = 975\ \Omega$$

$$R_{xx} = \rho_S \frac{L}{W} = 975\ \Omega$$

$$\rho_S = \frac{W}{L} 975\ \Omega = \frac{1.70}{2.65}\times 975 = 625 = \frac{1}{n_S q \mu_n}$$

$$\mu_n = \frac{1}{n_S q \rho_S} = \frac{1}{r_H \times 2.92\times 10^{16}\times 1.6\times 10^{-19}\times 625}$$

$$r_H \mu_n = \mu_H = \frac{1}{2.92\times 10^{16}\times 1.6\times 10^{-19}\times 625} = 0.3\,\text{cm}^2/\text{V-s}$$

$$\boxed{\mu_H = 0.3\,\text{cm}^2/\text{V-s}}$$

Not the mobility, but the "Hall mobility."

5d) If the scattering time is 1ps, find the magnetic field for which this classical analysis of Hall effect is no longer valid?

Solution:

We require:

$$\boxed{\omega_c \tau << 1}$$

$$\boxed{\frac{qB_z}{m^*}\tau << 1} \qquad \text{(i)}$$

$$\boxed{\mu_n = \frac{q\tau}{m^*}} \qquad \boxed{\tau = \frac{m^*\mu_n}{q}} \qquad \text{(ii)}$$

With (ii), (i) becomes

$$\boxed{\frac{qB_z}{m^*}\frac{m^*\mu_n}{q} << 1 \rightarrow B_z\mu_n << 1}$$

Using the Hall mobility as an estimate of the real mobility

$$\boxed{B_z\mu_n = 0.5 \times 0.3 = 0.15 << 1}$$

so this experiment is in the low B-field regime. The largest the B-field can be to be in the low field regime is

$$\boxed{B_z = \frac{1}{\mu_n} = \frac{1}{0.3} = 3.33\ \mathrm{T}}$$

$$\boxed{\boxed{B_z = 3.33\ \mathrm{T}}}$$

6) Contact resistances are important. They can complicate measurements of semiconductor transport parameters, and they can degrade device performance. The constant resistance is specified by the interfacial contact resistivity, ρ_C, in $\Omega\text{-cm}^2$. A very good value is $\rho_C \approx 10^{-8}\Omega\text{-cm}^2$. Consider n+ Si at room temperature and doped to $N_D = 10^{20}\ \text{cm}^{-3}$. What is the lower limit to ρ_C? (Assume a fully degenerate semiconductor and use appropriate effective masses for the conduction band of Si.)

Solution:

The lower limit resistance must be the ballistic contact resistance:

$$R_B = \frac{1}{G_B} = \frac{1}{\left(2q^2/h\right)\langle\langle T\rangle\rangle\langle M\rangle}$$

Assuming that one-half of the ballistic resistance is associated with each of the two contacts:

$$\rho_C = \frac{R_B A}{2} = \frac{h}{4q^2}\frac{1}{\langle\langle \mathcal{T}\rangle\rangle\langle M/A\rangle}$$

Assume a strongly degenerate semiconductor:

$$\rho_C = \frac{h}{4q^2}\frac{1}{\mathcal{T}\left(E_F\right)M\left(E_F\right)/A}$$

The lower limit occurs when the transmission is one

$$\rho_C^{\min} = \frac{h}{4q^2}\frac{1}{M\left(E_F\right)/A}$$

Need to find the Fermi level. Recall that at 0 K,

$$n_0 = \int_{E_C}^{E_F} D_{3D}\left(E\right)dE$$

$$D_{3D}\left(E\right) = \frac{\left(m_{DOS}^*\right)^{3/2}\sqrt{2\left(E-E_C\right)}}{\pi^2\hbar^3}$$

$$n_0 = \int_{E_C}^{E_F} D_{3D}\left(E\right)dE = \int_{E_C}^{E_F}\frac{\left(m_{DOS}^*\right)^{3/2}\sqrt{2\left(E-E_C\right)}}{\pi^2\hbar^3}dE = \frac{\sqrt{2}\left(m_{DOS}^*\right)^{3/2}}{\pi^2\hbar^3}\int_{E_C}^{E_F}\left(E-E_C\right)^{1/2}dE$$

$$n_0 = \frac{2\sqrt{2}\left(m_{DOS}^*\right)^{3/2}}{3\pi^2\hbar^3}\left(E_F-E_C\right)^{3/2}$$

$$\left(E_F-E_C\right) = \frac{1}{m_{DOS}^*}\left(\frac{3\pi^2\hbar^3}{2\sqrt{2}}\right)^{2/3}\left(n_0\right)^{2/3}$$

$$M_{3\mathrm{D}}\left(E_F\right) = \frac{m_{DOM}^*}{2\pi\hbar^2}\left(E_F-E_C\right)$$

$$M_{3\mathrm{D}}\left(E_F\right)=\frac{m^*_{DOM}}{m^*_{DOS}}\frac{1}{2\pi\hbar^2}\left(\frac{3\pi^2\hbar^3}{2\sqrt{2}}\right)^{2/3}\left(n_0\right)^{2/3}=\frac{m^*_{DOM}}{m^*_{DOS}}\left(\frac{3\sqrt{\pi}}{8}\right)^{2/3}n_0^{2/3}$$

For Si, we have to consider the ellipsoidal bandstructure:

$$m^*_{DOS}=\left(6\right)^{2/3}\left(m_t^2 m_\ell\right)^{1/3}=1.06m_0$$
$$m^*_{DOM}=2m_t^*+4\sqrt{m_t^* m_\ell^*}=2.04m_0$$

(See: Jeong, Changwook; Kim, Raseong; Luisier, Mathieu; Datta, Supriyo; and Lundstrom, Mark S., "On Landauer versus Boltzmann and full band versus effective mass evaluation of thermoelectric transport coefficients," *J. Appl. Phys.*, **107**, 023707, 2010.)

$$M_{3\mathrm{D}}\left(E_F\right)=\frac{m^*_{DOM}}{m^*_{DOS}}\left(\frac{3\sqrt{\pi}}{8}\right)^{2/3}n_0^{2/3}=\frac{2.04}{1.06}\left(0.762\right)\left(10^{20}\right)^{2/3}=3.16\times10^{13}\ \mathrm{cm}^{-2}$$

and finally:

$$\rho_C^{\min}=\frac{h}{4q^2}\frac{1}{M\left(E_F\right)/A}=6.48\times10^3\,\frac{1}{3.16\times10^{13}}=2.1\times10^{-10}\ \Omega\text{-cm}^2$$

$$\boxed{\rho_C^{\min}=2.1\times10^{-10}\ \Omega\text{-cm}^2}$$

So even for a very good interfacial contact resistivity, $\mathcal{T}\left(E_F\right)\approx 0.01$

For more on this topic, see:

J. Maassen, C. Jeong, A. Baraskar, M. Rodwell, and M. Lundstrom, “Full band calculations of the intrinsic lower limit of contact resistivity,” *Appl. Phys. Lett.*, Vol. 102, 2013. DOI: 10.1063/1.4798238